

# Modular representations of the special linear groups with small weight multiplicities

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## Abstract

We classify irreducible representations of the special linear groups in positive characteristic with small weight multiplicities with respect to the group rank and give estimates for the maximal weight multiplicities. We also classify inductive systems of representations with totally bounded weight multiplicities for the natural embeddings of the classical groups.

## 1 Introduction

In what follows  $K$  is an algebraically closed field of characteristic  $p > 0$ ;  $G_n$  is a classical algebraic group of rank  $n$  over  $K$ ;  $\text{Irr } G_n$  is the set of all rational irreducible representations (or simple modules) of  $G_n$  up to equivalence,  $\text{Irr}_p G_n \subset \text{Irr } G_n$  is the subset of  $p$ -restricted ones;  $M^*$  is the dual of a module  $M$ ;  $\text{Irr } M \subset \text{Irr } G_n$  is the set of composition factors of a module  $M$  (disregarding the multiplicities),  $\omega(M)$  is the highest weight of a simple module  $M$ ;  $L(\omega)$  is the simple  $G_n$ -module with highest weight  $\omega$ ;  $\omega_1^n, \dots, \omega_n^n$  (or, simply,  $\omega_1, \dots, \omega_n$ ) are the fundamental weights of  $G_n$ ;  $\omega_0^n = \omega_{n+1}^n = 0$  by convention;  $\text{Fr}$  is the Frobenius morphism of  $G_n$  associated with raising the elements of  $K$  to the  $p$ th power;  $M^{[k]}$  denotes a  $G$ -module  $M$  twisted by the  $k$ th power of  $\text{Fr}$ . A weight  $\sum_{i=1}^n a_i \omega_i^n$  is  $p$ -restricted if all  $a_i < p$ . For any dominant weight  $\omega$  of  $G_n$  denote by  $\delta(\omega)$  the value of  $\omega$  on the maximal root of the root system of  $G_n$ . For a simple module  $M \cong L(\omega)$  put  $\delta(M) = \delta(\omega)$ . Note that  $G_k$  is a natural subgroup of  $G_n$  for  $k < n$ . For a  $G_n$ -module  $M$  we denote by  $M \downarrow G_k$  the restriction of  $M$  to  $G_k$  and define  $\text{Irr}_k M = \text{Irr}(M \downarrow G_k)$ . By the

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weight degree of a module  $M$  we mean the maximal dimension of the weight subspaces in  $M$ , i.e.

$$\text{wdeg } M = \max_{\mu \in \Lambda(M)} \dim M^\mu$$

where  $\Lambda(M)$  is the set of weights of  $M$ . In particular, we say that  $M$  has a small weight degree if  $\text{wdeg } M$  is small with respect to  $n$ . Note that  $\text{wdeg } M = \text{wdeg } M^*$ .

For the classical algebraic groups modular representations of weight degree 1 were classified in [22, 28]. To state the result, first define the following sets of weights of the group  $G_n$ .

$$\begin{aligned} \Omega_p(G_n) &= \begin{cases} \{0, \omega_k^n, (p-1-a)\omega_k^n + a\omega_{k+1}^n \mid 0 \leq k \leq n, 0 \leq a \leq p-1\}, & G_n = A_n(K), \\ \{0, \omega_1^n, \omega_n^n\}, & G_n = B_n(K), \\ \{0, \omega_1^n, \frac{p-1}{2}\omega_n^n, \omega_{n-1}^n + \frac{p-3}{2}\omega_n^n\}, & G_n = C_n(K), \\ \{0, \omega_1^n, \omega_{n-1}^n, \omega_n^n\}, & G_n = D_n(K), \end{cases} \\ \Omega(G_n) &= \left\{ \sum_{j=0}^s p^j \lambda_j \mid s \geq 0, \lambda_j \in \Omega_p(G_n) \right\}. \end{aligned}$$

**Theorem 1.1** ([22, 28]) *Let  $G_n$  be a classical algebraic group of rank  $n \geq 4$  and let  $M$  be a rational simple  $G_n$ -module. Assume  $p > 2$  for  $G = B_n(K)$  or  $D_n(K)$ . Then  $\text{wdeg } M = 1$  if and only if  $\omega(M) \in \Omega(G_n)$ .*

Note that  $M$  is  $p$ -restricted with  $\text{wdeg } M = 1$  if and only if  $\omega(M) \in \Omega_p(G_n)$  and the  $A_n(K)$ -modules  $L((p-1-a)\omega_k + a\omega_{k+1})$  are truncated symmetric powers of the natural module [29]. Thus, the only  $p$ -restricted modules of weight degree 1 for type  $A$  are the fundamental modules and truncated symmetric powers of the natural module.

In this paper we classify irreducible representations of the special linear groups of small weight degree. For other classical groups, the authors previously proved that no modules  $M$  exist with  $1 < \text{wdeg } M < n-7$ .

**Theorem 1.2** ([1, Theorem 1.1], [20, Theorem 1], [21, Theorem 1]) *Let  $n \geq 8$  and let  $G_n = B_n(K)$ ,  $C_n(K)$  or  $D_n(K)$ . Let  $M$  be a rational simple  $G_n$ -module with  $\omega(M) \notin \Omega(G_n)$ . Suppose that  $p > 2$  for  $G_n = B_n(K)$  or  $C_n(K)$ . Then  $\text{wdeg } M \geq n-4-[n]_4$  where  $[n]_4$  is the residue of  $n$  modulo 4. In particular,  $\text{wdeg } M \geq n-7$ .*

The main case ( $p > 2$  for  $G_n = B_n(K), D_n(K)$  and  $p > 7$  for  $G_n = C_n(K)$ ) was settled in [1]; [20] deals with type  $D$  for  $p = 2$ ; and [21] gives a new proof for type  $C$  for all  $p$ , with new exceptional series of modules with  $\text{wdeg} = 2^s$  being added for  $p = 2$ , see [21, Theorem 2] for details (note that  $B_n(K) \cong C_n(K)$  for  $p = 2$ ).

Now assume that  $G_n = A_n(K)$ . Denote by  $V_n$  the natural module for  $G_n$ . Let  $M \in \text{Irr } G_n$  and  $\omega(M) = a_1\omega_1 + \dots + a_n\omega_n$ . Recall that  $\omega(M^*) = a_n\omega_1 + a_{n-1}\omega_2 + \dots + a_1\omega_n$ . Define the *polynomial degree* of  $M$  as the polynomial degree of the corresponding polynomial representation of  $GL_{n+1}(K)$ , i.e.

$$\text{pdeg } M = \sum_{k=1}^n ka_k. \quad (1)$$

Note that every simple module of polynomial degree  $d$  can be obtained as a composition factor of the  $d$ th tensor power  $V_n^{\otimes d}$ . More exactly, we have the following. Set

$$\mathcal{L}_n^d = \cup_{j \leq d} \text{Irr } V_n^{\otimes j}, \quad \mathcal{R}_n^d = \cup_{j \leq d} \text{Irr}(V_n^*)^{\otimes j}. \quad (2)$$

Then  $\mathcal{L}_n^d = \{M \in \text{Irr } G_n \mid \text{pdeg } M \leq d\}$  and  $\mathcal{R}_n^d = \{M \in \text{Irr } G_n \mid \text{pdeg } M^* \leq d\}$  (Proposition 4.2). For  $d \leq n$ , it is not difficult to see that  $\text{wdeg } V_n^{\otimes d} = d!$  (Lemma 4.4). This means that modules of small polynomial degree  $d$  (with, say,  $d! < n$ ) have small weight degree ( $< n$ ), which gives many more small weight degree modules for type  $A$  in addition to those described in Theorem 1.1. This makes situation more difficult than in the case of other classical groups, especially for non  $p$ -restricted modules. Our first main result describes  $p$ -restricted irreducible representations of the special linear groups of small weight degree.

**Theorem 1.3** *Let  $M \in \text{Irr}_p A_n(K)$  and  $d = \min\{\text{pdeg } M, \text{pdeg } M^*\}$ . Assume  $\omega(M) \notin \Omega_p(A_n(K))$ . Then the following holds.*

(i) *If  $n \geq 16$  and  $d > n$ , then*

$$\text{wdeg } M > \sqrt{n}/p - 1.$$

(ii) *If  $d \leq n$ , then*

$$d - 2 \leq \text{wdeg } M \leq d!.$$

*Moreover,  $M \cong L(a_1\omega_1 + \dots + a_d\omega_d)$  or  $L(a_d\omega_{n-d+1} + \dots + a_1\omega_n)$  with  $a_1 + 2a_2 + \dots + da_d = d$ , and  $\text{wdeg } M$  is determined by the sequence  $(a_1, \dots, a_d)$  only and does not depend on  $n$ .*

*In particular, if  $n > 16$  and  $\text{wdeg } M \leq \sqrt{n}/p - 1$ , then  $M$  is as in part (ii) with  $d \leq \sqrt{n}/p + 1$ .*

The  $\sqrt{n}/p - 1$  estimate in part (i) was obtained by applying Schur functor. It is a quick and rough estimate and can probably be improved if one uses a more thorough analysis, similar to that of [1]. One should expect something close to  $n$ , as in Theorem 1.2. Unfortunately, this seems to be very difficult to obtain at the moment as too many modules of small weight degree exist for type  $A$  and the methods used in [1] fail to work. But our estimate is good enough to identify the modules with small weight degree and get a full classification of the inductive systems of representations for  $A_\infty$  with bounded weight multiplicities (see below).

Let  $M \in \text{Irr } G_n$ . Assume that  $\omega(M) = \sum_{k=0}^s p^k \lambda_k$  with  $p$ -restricted weights  $\lambda_k$  of  $G_n$ . Put  $M_k = L(\lambda_k)$ . By the Steinberg tensor product theorem [25],

$$M \cong \otimes_{k=0}^s M_k^{[k]}. \quad (3)$$

It is obvious that  $\text{wdeg } M \geq \text{wdeg } M_0 \dots \text{wdeg } M_s$  (Lemma 2.16). Therefore, the question of describing non  $p$ -restricted  $G_n$ -modules of small weight degree is essentially reduced to combining various Frobenius twists of  $p$ -restricted modules of small weight degree and making sure that the weight degree does not become too large (see Corollary 4.8, Theorem 4.10 and Proposition 4.11).

Note that the results above can be considered as a modular analogue of the following problem solved by Mathieu [18]: describe all infinite dimensional weight modules with bounded weight multiplicities for a finite dimensional simple Lie algebra over  $\mathbb{C}$ . Some particular cases, including so-called completely pointed modules (i.e. with one dimensional weight spaces) were previously considered in [5, 6, 9]. It is interesting to note that by specializing  $p$  to 0 in the weights in the set  $\Omega_p(G_n)$  we get highest weights of completely pointed modules (e.g.  $(-1-a)\omega_k + a\omega_{k+1}$  for type  $A_n$  and  $\omega_{n-1} - \frac{3}{2}\omega_n$  and  $-\frac{1}{2}\omega_n$  for type  $C_n$ ).

Estimates of weight multiplicities obtained above can be used for recognizing linear groups containing matrices with small eigenvalue multiplicities. Indeed, it occurs that only for some special classes of representations of simple classical algebraic groups, their images can contain matrices all whose eigenvalue multiplicities are small enough with respect to the group rank.

In the final section of the paper we classify inductive systems of representations with bounded weight multiplicities for the naturally embedded classical groups. Let

$$\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_n \subset \cdots \quad (4)$$

be a chain of algebraic groups  $\Gamma_n$  over  $K$  and let  $\Phi_n$ ,  $n \in \mathbb{N}$ , be a nonempty finite subset of  $\text{Irr } \Gamma_n$ , for each  $n$ . Recall that the system  $\Phi = \{\Phi_n \mid n \in \mathbb{N}\}$  is called an *inductive system* of representations (or modules) for (4) if

$$\bigcup_{\phi \in \Phi_{n+1}} \text{Irr}(\phi \downarrow \Gamma_n) = \Phi_n$$

for all  $n \in \mathbb{N}$ , where  $\phi \downarrow \Gamma_n$  is the restriction of  $\phi$  to  $\Gamma_n$ . Inductive systems have been introduced by Zalesskii in [26]. They can be considered as an asymptotic version of the branching rules for the embeddings (4). Moreover, inductive systems can be applied to the study of ideals in group algebras of locally finite groups. It is proved in [27] that there exists a bijective correspondence between the inductive systems for a locally finite group and the semiprimitive ideals of the corresponding group algebra. So far we know little about the structure of inductive systems. Minimal and minimal nontrivial inductive systems of modular representations for natural embeddings of algebraic and finite groups of type  $A_n$  were classified in [3]. For other classical groups the question on the minimal inductive systems seems substantially more difficult. For natural embeddings of symplectic groups in positive characteristic examples of such systems that have no analogues in the characteristic 0 case were constructed in [28] and [2]. In this paper we consider only inductive systems for the natural embeddings of groups  $G_n$  introduced above.

**Definition 1.4** Let  $\Phi$  be an inductive system of representations. We say that  $\Phi$  is a *BWM-system* (bounded weight multiplicities system) if there exists  $m \in \mathbb{N}$  such that  $\text{wdeg } \phi \leq m$  for all  $\phi \in \Phi_n$  and all  $n$ . For a BWM-system  $\Phi$  we define  $\text{wdeg } \Phi = \max_{\phi \in \Phi} \text{wdeg } \phi$ .

In the final section we classify all BWM-systems for all four types of classical groups. To state the main results, we need to introduce some notation. Let  $T \subset \mathbb{N}$  be infinite. Assume that  $R_t \subset \text{Irr } G_t$  is nonempty for each  $t \in T$  and that there exists  $k \in \mathbb{N}$  such that  $\delta(M) < k$  for all  $M \in R_t$  and for all  $t$ . Denote by  $\Pi_n$  the set of all  $G_n$ -modules  $Q$  such that  $Q$  is a composition factor of the restriction  $Y \downarrow G_n$  for some  $t > n$ ,  $t \in T$ , and  $Y \in R_t$ . Assume that  $R_t \subset \Pi_t$  for all  $t$ . Then it is easy to check that  $\Pi = \{\Pi_n \mid n = 1, 2, \dots\}$  is an inductive system for the groups  $G_n$  (see [4]). We will write  $\Pi = \langle R_t \mid t \in T \rangle$  and call  $\Pi$  the *inductive system generated by  $R_t$* . If every  $R_t$  consists of a single module  $Y_t$ , we use a simplified notation  $\Pi = \langle Y_t \mid t \in T \rangle$ . Let  $\Phi$  be an inductive system. We say that  $\Phi$  is a *p-restrictedly generated system* if  $\Phi = \langle \Lambda_t \mid t \in T \rangle$  with  $\Lambda_t \subset \text{Irr}_p G_t$  for all  $t \in T$ .

For arbitrary inductive systems  $\Phi$  and  $\Psi$  define the collections  $\text{Fr}(\Phi)$  and  $\Phi \otimes \Psi$  in a natural way:

$$\text{Fr}(\Phi)_n = \{\phi^{[1]} \mid \phi \in \Phi_n\},$$

$$(\Phi \otimes \Psi)_n = \bigcup_{\phi \in \Phi_n, \psi \in \Psi_n} \text{Irr}(\phi \otimes \psi).$$

By Lemma 3.2,  $\text{Fr}(\Phi)$  and  $\Phi \otimes \Psi$  are inductive systems. The union of inductive systems  $\Phi$  and  $\Psi$  and the inclusion relation for such systems are defined in a natural way. An inductive system  $\mathcal{T}$  is called *decomposable* if  $\mathcal{T}$  is the union of inductive systems  $\Phi$  and  $\Psi$  that do not coincide with  $\mathcal{T}$ , and *indecomposable* otherwise. For an inductive system  $\Phi$  put

$$\delta(\Phi_n) = \max\{\delta(\phi) \mid \phi \in \Phi_n\}.$$

Then  $\delta(\Phi_n)$  does not depend on  $n$  (Lemma 3.1), so we can define  $\delta(\Phi)$  as  $\delta(\Phi_n)$ .

In Section 3 we prove the following analogue of Steinberg product theorem for inductive systems.

**Theorem 1.5** *Let  $\Phi$  be an indecomposable inductive system. Then there exist  $p$ -restrictedly generated inductive systems  $\Phi^j$ ,  $0 \leq j \leq k$ , such that  $\Phi = \bigotimes_{j=0}^k \text{Fr}^j(\Phi^j)$ .*

Now assume that  $G_n = A_n(K)$ . Recall the sets  $\mathcal{L}_n^d$  and  $\mathcal{R}_n^d$  defined in (2). Lemma 3.6 implies that  $\mathcal{L}^d = \{\mathcal{L}_n^d \mid n \in \mathbb{N}\}$  and  $\mathcal{R}^d = \{\mathcal{R}_n^d \mid n \in \mathbb{N}\}$  are inductive systems. Note that  $\mathcal{L}_n^1 = \{0, V_n\}$ . Set

$$\mathcal{F}_n = \{L(\omega_0^n), L(\omega_1^n), \dots, L(\omega_n^n)\}, \quad (5)$$

$$\mathcal{T}_n = \{L((p-a-1)\omega_i^n + a\omega_{i+1}^n) \mid 0 \leq a < p, \quad 0 \leq i \leq n\} \quad (6)$$

( $\omega_{n+1}^n$  is treated as 0). By Lemma 3.6,  $\mathcal{F} = \{\mathcal{F}_n \mid n \in \mathbb{N}\}$  and  $\mathcal{T} = \{\mathcal{T}_n \mid n \in \mathbb{N}\}$  are inductive systems. Note that the representations of  $\mathcal{T}$  are realized exactly in the truncated symmetric powers of the natural module.

Let  $d \in \mathbb{N}$ . Fix any integers  $a_i \geq 0$  for  $0 \leq i \leq d$ . Let  $M_{n,L}(a_1, \dots, a_d)$  be a simple  $G_n$ -module with highest weight  $a_1\omega_1 + \dots + a_d\omega_d$  and  $M_{n,R}(a_1, \dots, a_d)$  be a simple  $G_n$ -module with highest weight  $a_d\omega_{n-d+1} + \dots + a_1\omega_n$ . Set

$$C_L(a_1, \dots, a_d) = \langle M_{n,L}(a_1, \dots, a_d) \mid n \geq d \rangle, \quad (7)$$

$$C_R(a_1, \dots, a_d) = \langle M_{n,R}(a_1, \dots, a_d) \mid n \geq d \rangle. \quad (8)$$

**Theorem 1.6** *Let  $G_n = A_n(K)$ . Assume that  $\Phi$  is a  $p$ -restrictedly generated indecomposable BWM-system. Then  $\Phi = \mathcal{F}, \mathcal{T}, C_L(a_1, \dots, a_d)$  or  $C_R(a_1, \dots, a_d)$  for some integers  $a_1, \dots, a_d < p$ .*

Let  $\Phi$  be an inductive system. Assume

$$\Phi = \bigotimes_{k=0}^s \text{Fr}^k(\Phi^k), \quad (9)$$

where  $\Phi^k$  are  $p$ -restrictedly generated systems. We say that  $\Phi$  is *special* if each  $\Phi^k$  is equal to one of the systems  $C_L(a_1, \dots, a_d)$ ,  $C_R(a_1, \dots, a_d)$ ,  $\mathcal{F}$ , or  $\mathcal{T}$ .

Let  $\Phi$  be special. Then for every  $k$ , either  $\Phi^k = \mathcal{F}, \mathcal{T}$  or there exists  $d$  such that  $\Phi^k \subset \mathcal{L}^d$  or  $\mathcal{R}^d$ . Therefore,  $\Phi$  can be represented in the form

$$\Phi = \Psi^0 \otimes \dots \otimes \Psi^l, \quad (10)$$

with

$$\Psi^f = \bigotimes_{k=i_{f-1}+1}^{i_f} \text{Fr}^k(\Phi^k), \quad (11)$$

where the indices  $i_f$ ,  $0 \leq f \leq l$ , satisfy the following:  $i_{-1} = -1$  and for each  $f$ , either all  $\Phi^k \subset \mathcal{L}^d$  for  $i_{f-1} + 1 \leq k \leq i_f$  or all  $\Phi^k \subset \mathcal{R}^d$  for  $i_{f-1} + 1 \leq k \leq i_f$ , or  $i_{f-1} + 1 = k = i_f$  and  $\Phi^k = \mathcal{F}$  or  $\mathcal{T}$ . Fix minimal  $l$  with this property. Then the systems  $\Psi^f$  are uniquely determined.

**Theorem 1.7** *Let  $G = A_n(K)$ . Indecomposable BWM-systems are exhausted by special inductive systems with the following property  $\delta(\Psi^f) < p^{i_f+1}$  for all  $\Psi^f$  with  $f < l$  ( $i_f$  are such as in (11)). An arbitrary BWM-system is a finite union of indecomposable ones.*

Theorem 1.2 allows us to find the BWM-systems for the remaining series of classical groups. Put

$$\mathcal{S}_n = \begin{cases} \{L(\omega_n^n)\} & \text{for } G_n = B_n(K), \\ \{L(\omega_{n-1}^n), L(\omega_n^n)\} & \text{for } G_n = D_n(K), \\ \{L(\frac{p-1}{2}\omega_n^n), L(\omega_{n-1}^n + \frac{p-3}{2}\omega_n^n)\} & \text{for } G_n = C_n(K) \end{cases}$$

and  $\mathcal{L}_n = \{L(0), L(\omega_1^n)\}$ . Lemma 3.6 implies that  $\mathcal{S} = \{\mathcal{S}_n\}$  and  $\mathcal{L} = \{\mathcal{L}_n\}$  are inductive systems. Obviously, the collection  $\mathcal{O} = \{\mathcal{O}_n\}$  with  $\mathcal{O}_n = \{L(0)\}$  is an inductive system for all types.

**Theorem 1.8** *Let  $G_n = B_n(K), C_n(K)$  or  $D_n(K)$ , and let  $p > 2$ . Set  $\mathcal{P} = \{\mathcal{O}, \mathcal{L}, \mathcal{S}\}$ . An indecomposable inductive system  $\Phi$  is a BWM-system if and only if  $\Phi = \otimes_{j=0}^s \text{Fr}^j(\Phi^j)$ , where  $\Phi^j \in \mathcal{P}$ . BWM-systems are finite unions of indecomposable ones and consist of modules with one dimensional weight spaces.*

## 2 Notation and preliminaries

Let  $\mathbb{N}$  and  $\mathbb{Z}_{\geq 0}$  be the sets of positive and nonnegative integers, respectively. For a simple algebraic group  $G$  over  $K$  the symbol  $\Lambda(G)$  denotes the set of all weights of  $G$ ,  $R(G)$  is the set of all roots of  $G$ ,  $R^+(G)$  is the set of all positive roots of  $G$  (with respect to a fixed maximal torus and a fixed base of  $R(G)$ ); we identify  $\Lambda(A_1(K))$  with  $\mathbb{Z}$ ;  $\langle \lambda, \alpha \rangle$  is the value of a weight  $\lambda \in \Lambda(G)$  on a root  $\alpha \in R(G)$ , and  $\text{Irr } G$  is defined as for groups  $G_n$ . Throughout the text  $\Lambda(M)$  is the set of all weights of a  $G$ -module  $M$ ,  $\omega(M)$  is the highest weight of a simple module  $M$ ,  $L(\omega)$  and  $V(\omega)$  are the simple  $G$ -module with the highest weight  $\omega$  and the Weyl module for  $\omega$ , respectively. For a  $G$ -module  $M$  denote by  $v^+$  a nonzero highest weight vector of  $M$  and by  $M^\mu$  the weight space in  $M$  of a weight  $\mu$ . For  $\alpha \in R(G)$ ,  $t \in K$ ,  $k \in \mathbb{Z}_{\geq 0}$  the symbols  $X_\alpha$ ,  $\mathcal{X}_\alpha$ , and  $X_{\alpha,k}$  denote the root elements of the Lie algebra of  $G$ , the root subgroup of  $G$  associated with  $\alpha$ , and the element of the hyperalgebra associated with the pair  $(\alpha, k)$ , respectively. For  $k < p$  one has  $X_{\alpha,k} = (X_\alpha)^k/k!$ . The subgroup of a group  $G$  generated by subgroups  $\Gamma_1, \dots, \Gamma_i$  and the subspace of a linear space  $L$  spanned by vectors  $v_1, \dots, v_i$  are denoted by  $\langle \Gamma_1, \dots, \Gamma_i \rangle$  and  $\langle v_1, \dots, v_i \rangle$ , respectively. For  $\beta_1, \dots, \beta_j \in R^+(G)$  put

$$G(\beta_1, \dots, \beta_j) = \langle \mathcal{X}_{\beta_1}, \dots, \mathcal{X}_{\beta_j}, \mathcal{X}_{-\beta_1}, \dots, \mathcal{X}_{-\beta_j} \rangle.$$

In all cases where subgroups of this form are considered, the roots  $\beta_1, \dots, \beta_j$  are chosen such that they constitute a base of the root system of  $G(\beta_1, \dots, \beta_j)$ . In this situation the fundamental weights of  $G(\beta_1, \dots, \beta_j)$  are determined with respect to this base. If  $H = G(\beta_1, \dots, \beta_k) \subset G$  and  $\omega \in \Lambda(G)$ , then  $\omega \downarrow H$  is the restriction of  $\omega$  to  $H$ . For a

$G$ -module  $M$  and a weight vector  $v \in M$  we denote the weight of  $v$  with respect to a subgroup  $H \subset G$  by  $\omega_H(v)$ . Set  $\omega(v) = \omega_G(v)$ .

We fix a base  $\alpha_1, \dots, \alpha_n$  of  $R(G_n)$  (labeled as in [7]), the fundamental weights are considered with respect to this base. In what follows  $\varepsilon_i$ ,  $1 \leq i \leq n+1$  for  $G = A_n(K)$  and  $1 \leq i \leq n$  otherwise, are weights of the natural realization of  $G_n$ , their labeling is standard and corresponds to [7, Ch. VIII, §13]. Set  $X_{\pm i} = X_{\pm \alpha_i}$  and define  $\mathcal{X}_{\pm i}$  and  $X_{\pm i, k}$  similarly. Put  $G(i_1, \dots, i_j) = G(\alpha_{i_1}, \dots, \alpha_{i_j})$  and  $b_i(\mu) = \langle \mu, \alpha_i \rangle$ .

**Lemma 2.1 (Seitz [22, 1.5])** *Let  $M$  be a  $G$ -module and  $v \in M \setminus \{0\}$  be a vector of weight  $\lambda$ . Assume that  $\langle \lambda, \alpha \rangle = m < p$  for  $\alpha \in R(G)$  and that  $\mathcal{X}_\alpha$  fixes  $v$ . Then  $X_{-\alpha, k}v \neq 0$  for  $0 \leq k \leq m$ .*

**Lemma 2.2** *Let  $G = A_1(K)$ , and let  $M$  be an indecomposable  $G$ -module of highest weight  $p+b$  with  $0 \leq b < p-1$ . Assume that  $X_{-\alpha}^{b+1}v^+ \neq 0$  for  $\alpha \in R^+(G)$ . Then  $M \cong V(p+b)$ .*

*Proof.* Set  $b_1 = p+b$  and  $b_2 = p-b-2$ . By the universal property of the Weyl module [15, Part II, Lemma 2.13b)],  $M$  is a quotient of  $V(b_1)$ . It follows from [8] (and can be easily deduced from the weight structure of  $V(b_1)$ ) that  $V(b_1)$  has two composition factors:  $L(b_1)$  and  $L(b_2)$ . The Steinberg tensor product theorem [25] (see (3)) forces that  $b_2 \notin \Lambda(L(b_1))$ . However,  $b_2 \in \Lambda(M)$  as  $X_{-\alpha}^{b+1}v^+ \neq 0$ . This implies that  $M \not\cong L(b_1)$  and completes the proof.  $\square$

**Theorem 2.3 (Jantzen [14], Smith [23])** *Let  $H = G(i_1, \dots, i_j) \subset G_n$ . Then  $KHv^+ \subset L(\omega)$  is an irreducible  $H$ -module with highest weight  $\omega_H(v^+)$  and a direct summand of the  $H$ -module  $L(\omega)$ .*

Call  $KHv^+$  in the previous theorem the *Smith factor* of  $L(\omega)$  (with respect to  $H$ ).

Let  $H = G(\beta_1, \dots, \beta_j) \subset G$  and  $M$  be a  $G$ -module. Put  $U^+(H) = \langle \mathcal{X}_\alpha \mid \alpha \in R^+(H) \rangle$ .

**Definition 2.4** A vector  $v \in M$  is called *primitive* with respect to  $H$  (or a *primitive  $H$ -vector*) if  $v$  is a nonzero weight vector and  $U^+(H)$  fixes  $v$ .

**Definition 2.5** Let  $\omega = a_1\omega_1 + \dots + a_n\omega_n$  be a dominant weight for  $G_n$  and let  $M = L(\omega)$ . Put  $y_k = -\langle \alpha_{k-1}, \alpha_k \rangle$  and  $z_k = -\langle \alpha_{k+1}, \alpha_k \rangle$ . Suppose that the roots  $\alpha_t$  with  $t$  in the interval with the ends  $i$  and  $j$  ( $1 \leq i, j \leq n$ ) form a chain on the Dynkin diagram of  $G_n$ . Fix  $v^+$ . For an integer  $d$  with  $0 < d \leq a_j$  define the vector  $v(i, j, d)$  as follows. Put  $d_j = d$ . If  $i > j$ , put  $d_k = a_k + d_{k-1}y_k$  for  $i \geq k > j$ . If  $i < j$ , set  $d_k = a_k + d_{k+1}z_k$  for  $i \leq k < j$ . Now define

$$v(i, j, d) = X_{-i, d_i} \dots X_{-k, d_k} \dots X_{-j, d} v^+.$$

**Lemma 2.6 ([24, Lemma 2.9])** *The vector  $v(i, j, d) \neq 0$  and  $X_{l, b}v(i, j, d) = 0$  for  $l \neq i$  and  $b > 0$ . Hence  $\mathcal{X}_l$  fixes  $v(i, j, d)$ .*

Recall the set of  $G_n$ -modules  $\mathcal{F}_n$  defined in (5).

**Lemma 2.7** *Let  $G_n = A_n(K)$ .*

$$(i) \text{ Irr}_{n-1} L(\omega_i^n) = \{L(\omega_{i-1}^{n-1}), L(\omega_i^{n-1})\}$$

$$(ii) \text{ Let } k < i \leq n - k + 1, M \in \text{Irr } G_n, \text{ and } \omega(M) = \omega_i^n. \text{ Then } \text{Irr}_k M = \mathcal{F}_k.$$

*Proof.* (i) One has  $L(\omega_i^n) = \wedge^i V_n$  [15, Part II, 2.15]. Let  $v_1, \dots, v_{n+1} \in V_n$  and  $\omega(v_i) = \varepsilon_i$ . One can assume that  $\varepsilon_{n+1} \downarrow G_{n-1} = 0$  and  $G_{n-1}$  fixes  $\langle v_1, \dots, v_n \rangle$  and  $v_{n+1}$ . Then the  $G_{n-1}$ -module  $\langle v_1, \dots, v_n \rangle$  is isomorphic to the natural  $G_{n-1}$ -module. Set

$$U_1 = \langle v_{k_1} \wedge \dots \wedge v_{k_i} \mid 1 \leq k_1 < \dots < k_i < n+1 \rangle$$

and

$$U_2 = \langle v_{l_1} \wedge \dots \wedge v_{l_{i-1}} \wedge v_{n+1} \mid 1 \leq l_1 < \dots < l_{i-1} < n+1 \rangle.$$

Then  $\wedge^i V_n = U_1 \oplus U_2$ . One easily observes that  $G_{n-1}$  fixes  $U_1$  and  $U_2$ , the  $G_{n-1}$ -module  $U_1 \cong L(\omega_i^{n-1})$  and  $U_2 \cong L(\omega_{i-1}^{n-1})$ .

(ii) Put  $H_j = G(i-j+1, i-j+2, \dots, i-j+k)$  for  $1 \leq j \leq k$  and  $H_0 = G(1, \dots, k)$ . The subgroups  $H_j$  are conjugate to  $G_k$ . Restricting  $M$  to subgroups  $H_j$  and applying Theorem 2.3, we get that  $L(\omega_j^k) \in \text{Irr}(M \downarrow G_k)$  for all  $0 \leq j \leq k$ .  $\square$

**Lemma 2.8** ([29]) *Let  $G_n = A_n(K)$  and  $H = G(1, \dots, m, m+2, \dots, n) \subset G_n$ . Then*

$$L(c\omega_i^n + (p-1-c)\omega_{i+1}^n) \downarrow H =$$

$$= \oplus_{N(i,c)} L(c_1\omega_{i_1}^m + (p-1-c_1)\omega_{i_1+1}^m) \otimes L(c_2\omega_{i_2}^{n-m-1} + (p-1-c_2)\omega_{i_2+1}^{n-m-1})$$

with  $N(i, c) = \{(i_1, c_1), (i_2, c_2) \mid 0 \leq c_j < p, \quad 0 \leq (p-1)(i_1+1) - c_1 \leq (p-1)(m+1), \quad 0 \leq (p-1)(i_2+1) - c_2 \leq (p-1)(n-m), \quad (p-1)(i_1+i_2+2) - c_1 - c_2 = (p-1)i + p - 1 - c\}$ .

Recall the set of  $G_n$ -modules  $\mathcal{T}_n$  defined in (6).

**Corollary 2.9** *If  $G_n = A_n(K)$ ,  $k+1 \leq i < n-k$  and  $\omega = c\omega_i^n + (p-1-c)\omega_{i+1}^n$ , then  $\text{Irr}_k L(\omega) = \mathcal{T}_k$ .*

**Lemma 2.10** ([28, Theorem, part C]) *Let  $p > 2$ ,  $n > 1$ , and  $G_n = C_n(K)$ . Set  $M_1^n = L(\omega_{n-1}^n + \frac{p-3}{2}\omega_n^n) \in \text{Irr } G_n$  and  $M_2^n = L(\frac{p-1}{2}\omega_n^n) \in \text{Irr } G_n$ . Then  $\text{Irr}(M_j^n \downarrow G_{n-1}) = \{M_1^{n-1}, M_2^{n-1}\}$  for  $j = 1, 2$ .*

**Lemma 2.11** *Let  $G_n$  be a classical group. Then  $\text{Irr}_{n-1} V_n = \{0, V_{n-1}\}$ .*

*Proof.* This is obvious and well known.  $\square$

The following lemma is also well known, but we fail to find an explicit reference.

**Lemma 2.12** *If  $G_n = B_n(K)$  and  $n > 1$ , then  $\text{Irr}_{n-1}(L(\omega_n^n)) = \{L(\omega_{n-1}^{n-1})\}$ . For  $G_n = D_n(K)$  with  $n > 2$  one has  $\text{Irr}_{n-1}(L(\omega_n^n)) = \text{Irr}_{n-1}(L(\omega_{n-1}^n)) = \{L(\omega_{n-1}^{n-1}), L(\omega_{n-2}^{n-1})\}$ .*

*Proof.* For both types identify  $G_{n-1}$  with the subgroup  $G(2, \dots, n-1) \subset G_n$ . Let  $M$  be one of the modules in question. It is well known that  $\omega(M)$  is a microweight and hence  $\Lambda(M)$  coincides with the orbit of  $\omega(M)$  under the action of the Weyl group. Hence  $\Lambda(M) = \{(\pm\varepsilon_1 + \dots + \pm\varepsilon_n)/2\}$  with all possible combinations of the “plus” and “minus” signs for  $G_n = B_n(K)$ . If  $G_n = D_n(K)$ , then  $\Lambda(M)$  consists of all such weights with an odd or even number of the “minus” signs for  $M = L(\omega_{n-1}^n)$  or  $L(\omega_n^n)$ , respectively. Let  $M_+ \subset M$  ( $M_- \subset M$ ) be the sum of all weight subspaces  $M^\lambda$  with  $\lambda = \varepsilon_1/2 + \mu$  ( $\lambda = -\varepsilon_1/2 + \mu$ , respectively) where  $\mu$  is a linear combination of the weights  $\varepsilon_2, \dots, \varepsilon_n$ . Denote by  $\varepsilon_j^{n-1}$ ,  $1 \leq j \leq n-1$ , the weights of the natural  $G_{n-1}$  module  $V_{n-1}$  defined as



the weights  $\varepsilon_j$  for  $G_n$ . For  $2 \leq i \leq n$  one can identify the restriction of the weight  $\varepsilon_i$  to  $G_{n-1}$  with the weight  $\varepsilon_{i-1}^{n-1} \in \Lambda(G_{n-1})$ . Taking into account that for  $2 \leq i \leq n$  the roots  $\alpha_i$  are linear combinations of the weights  $\varepsilon_i$  with  $2 \leq i \leq n$ , one can observe that  $G_{n-1}$  fixes  $M_+$  and  $M_-$ . Analyzing the weight structure of these  $G_{n-1}$ -modules, we conclude that they are irreducible and have desired highest weights. This proves the lemma.  $\square$

Recall that  $\delta(\omega)$  is defined as the value of the weight  $\omega$  on the maximal root of the root system of  $G_n$  and  $\delta(L(\omega)) = \delta(\omega)$ .

**Lemma 2.13** *Let  $M \in \text{Irr } G_n$ , and let  $\alpha$  be a long root of  $G_n$ . Then  $\delta(M) = \max_{\lambda \in \Lambda(M)} \langle \lambda, \alpha \rangle$ .*

*Proof.* Denote by  $\alpha_{\max}$  the maximal root in  $R(G_n)$ . As  $\alpha_{\max}$  is a dominant weight,  $\langle \alpha_i, \alpha_{\max} \rangle \geq 0$ . This implies

$$\delta(M) = \langle \omega(M), \alpha_{\max} \rangle = \max_{\lambda \in \Lambda(M)} \langle \lambda, \alpha_{\max} \rangle.$$

Since the Weyl group acts transitively on the set of roots of the same length and  $\alpha_{\max}$  is long,  $\max_{\lambda \in \Lambda(M)} \langle \lambda, \alpha \rangle = \max_{\lambda \in \Lambda(M)} \langle \lambda, \alpha_{\max} \rangle$ , as required.  $\square$

**Corollary 2.14** *Let  $k < n$ ,  $M \in \text{Irr } G_n$  and  $N \in \text{Irr}_k M$ . Then  $\delta(N) \leq \delta(M)$ .*

**Proposition 2.15** *Let  $k < n$ ,  $M \in \text{Irr } G_n$ , and  $N \in \text{Irr}_k M$ . Then  $\text{wdeg } N \leq \text{wdeg } M$ .*

*Proof.* First assume that  $k = n - 1$ . Put  $\omega = \omega(M)$ . For every  $\lambda \in \Lambda(M)$  one has  $\lambda = \omega - \sum_{i=1}^n b_i(\lambda) \alpha_i$  with  $b_i(\lambda) \in \mathbb{Z}_{\geq 0}$ . For  $j \in \mathbb{Z}_{\geq 0}$  put

$$\Lambda_j = \{\lambda \in \Lambda(M) \mid b_1(\lambda) = j\}.$$

It is obvious that  $\Lambda_j \cap \Lambda_t = \emptyset$  for  $j \neq t$  and

$$\Lambda(M) = \Lambda_0 \cup \dots \cup \Lambda_l$$

for some  $l$ . Set

$$U_j = \oplus_{\lambda \in \Lambda_j} M^\lambda.$$

We can assume that  $G_{n-1} = G(2, \dots, n)$ . Then  $U_j$  are  $G_{n-1}$ -modules and  $M = U_0 \oplus \dots \oplus U_l$  as  $G_{n-1}$ -module. Hence  $N$  is realized in a composition factor of some module  $U_s$ . So  $\text{wdeg } N$  is not bigger than the maximal weight multiplicity of the  $G_{n-1}$ -module  $U_s$ . It remains to observe that the restrictions of distinct weights in  $\Lambda_s$  to  $G_{n-1}$  are distinct. Indeed, assume  $\mu, \nu \in \Lambda_s$  and  $\nu \neq \mu$ . Obviously  $b_1(\mu) = b_1(\nu)$ . Hence  $b_i(\mu) \neq b_i(\nu)$  for some  $i$  with  $2 \leq i \leq n$ . This yields that  $\mu \downarrow G_{n-1} \neq \nu \downarrow G_{n-1}$  and proves the lemma for  $k = n - 1$ . To complete the proof it remains to apply induction on  $n - k$ .  $\square$

The following lemma is obvious.

**Lemma 2.16** *Let  $M_1$  and  $M_2$  be  $G_n$ -modules. Then  $\text{wdeg } M_1^{[k_1]} \otimes M_2^{[k_2]} \geq \text{wdeg } M_1 \cdot \text{wdeg } M_2$ .*

### 3 The Steinberg tensor product theorem for inductive systems

Let  $\Phi = \{\Phi_n \mid n \in \mathbb{N}\}$  be an inductive system of representations for a classical series  $\{G_n\}$ . Put  $\delta(\Phi_n) = \{\max \delta(\omega) \mid L(\omega) \in \Phi_n\}$  and define  $\delta(\Phi) = \delta(\Phi_n)$  for  $n > 2$ . The following lemma shows that  $\delta(\Phi)$  is well defined.

**Lemma 3.1** *Assume that  $n \in \mathbb{N}$  and  $n > 2$  for  $G_n = B_n(K)$ . Then for an inductive system  $\Phi$  one has  $\delta(\Phi_{n+1}) = \delta(\Phi_n)$ .*

*Proof.* Fix any  $L(\lambda) \in \Phi_n$  and  $L(\mu) \in \Phi_{n+1}$  with  $\delta(\Phi_n) = \delta(\lambda)$  and  $\delta(\Phi_{n+1}) = \delta(\mu)$ . Put  $H = G(n-1) \cong A_1(K)$  for  $G_n = B_n(K)$  and  $H = G(n) \cong A_1(K)$  in the other cases. Recall that  $G_{n-1}$  is identified with the subgroup  $G(2, \dots, n)$ . Observe that  $H \subset G_1$  for  $G \neq B_n(K)$  and  $H \subset G_2$  for  $G = B_n(K)$ . Set  $I_l = \cup_{\phi \in \Phi_l} \text{Irr}(\phi \downarrow H)$ . It follows from the definition of an inductive system that  $I_n = I_{n+1}$ . Using Lemma 2.13, we get

$$\delta(\Phi_{n+1}) = \delta(\mu) = \max\{i \mid L(i) \in I_{n+1}\} = \max\{i \mid L(i) \in I_n\} = \delta(\lambda) = \delta(\Phi_n),$$

as required.  $\square$

For the groups of type  $A$  the previous lemma was proven in [3]. Note that for any dominant weight  $\omega = a_1\omega_1^n + \dots + a_n\omega_n^n$  of  $A_n(K)$  one has  $\delta(\omega) = a_1 + a_2 + \dots + a_n$ .

**Lemma 3.2** *Let  $\Phi$  and  $\Psi$  be inductive systems of representations. Then  $\text{Fr}(\Phi)$  and  $\Phi \otimes \Psi$  are inductive systems of representations.*

*Proof.* The claim on  $\text{Fr}(\Phi)$  follows immediately from the definition of an inductive system since for  $M \in \text{Irr } G_{n+1}$

$$\text{Irr}_n(M^{[1]}) = \{\mu^{[1]} \mid \mu \in \text{Irr}_n M\}.$$

Clearly the set  $(\Phi \otimes \Psi)_n$  is finite. It remains to note that the restriction of representations commutes with taking tensor products.  $\square$

**Definition 3.3** Let  $\Phi \supset \Psi$  be inductive systems of representations and the embedding is proper. Put  $\Xi_n = \Phi_n \setminus \Psi_n$ . Denote by  $D(\Phi, \Psi)$  the inductive system of representations generated by  $\Xi_n$  and call it the *difference* of two inductive systems.

It is shown in [4] that  $D(\Phi, \Psi)$  is well defined. Since the embedding is proper, for any  $n \in \mathbb{N}$  there exists  $n_0 > n$  such that the set  $\Xi_{n_0} \neq \emptyset$ . Hence  $D(\Phi, \Psi)_n \neq \emptyset$  for all  $n$ . One obviously has  $\Phi = \Psi \cup D(\Phi, \Psi)$ .

**Lemma 3.4** *Let  $\Phi$  be an indecomposable inductive system. Then for each two representations  $\phi$  and  $\psi \in \Phi_n$  there exist  $m > n$  and  $\xi \in \Phi_m$  such that  $\phi, \psi \in \text{Irr}_n \xi$ .*

*Proof.* For each  $l > n$  put  $P_l = \{\rho \in \Phi_l \mid \phi \in \text{Irr}_n \rho\}$ . It is clear that  $P_l \neq \emptyset$  and for any  $\mu \in P_l$  there exists  $\nu \in P_{l+1}$  such that  $\mu \in \text{Irr}_l \nu$ . Hence  $\mathcal{P} = \langle P_l \mid l > n \rangle$  is an inductive system. We claim that  $\mathcal{P} = \Phi$ . Indeed, otherwise  $D(\Phi, \mathcal{P}) = \Phi$  as  $\Phi$  is indecomposable. However,  $\phi \notin \text{Irr}_n \psi$  if  $\psi \in \Phi_l \setminus P_l$ , by the construction of  $P_l$ . This yields the contradiction as  $D(\Phi, \mathcal{P})$  is generated by the collection  $\Phi_l \setminus P_l$ . Hence  $\mathcal{P} = \Phi$ . So there exists  $m > n$  such that  $\psi \in \text{Irr}_n \rho$  for  $\rho \in P_m$ .  $\square$

**Corollary 3.5** *Let  $\Phi$  be an indecomposable inductive system and let  $\phi_1, \dots, \phi_l \in \Phi_n$ . Then there exist  $m > n$  and  $\xi \in \Phi_m$  such that  $\phi_1, \dots, \phi_l \in \text{Irr}_n \xi$ .*

Recall the collections  $\mathcal{L}^l, \mathcal{R}^l, \mathcal{F}, \mathcal{T}, \mathcal{S}$ , and  $\mathcal{L}$  defined in Introduction.

**Lemma 3.6** *The collections  $\mathcal{L}^l, \mathcal{R}^l$  ( $l \in \mathbb{N}$ ),  $\mathcal{F}$ , and  $\mathcal{T}$  are inductive systems of representations for the groups  $A_n(K)$ . The collections  $\mathcal{S}$  and  $\mathcal{L}$  are inductive systems for  $B_n(K)$ ,  $C_n(K)$ , and  $D_n(K)$ .*

*Proof.* First suppose that  $G_n = A_n(K)$ . By Lemma 2.7(i),  $\text{Irr}_{n-1} V_n = \{0, V_{n-1}\}$ . Hence  $\text{Irr} V_n^{\otimes l} \subset \text{Irr}_n V_{n+1}^{\otimes l}$ ,  $\text{Irr}_{n-1} V_n^{\otimes l} \subset \cup_{j \leq l} \text{Irr} V_{n-1}^{\otimes j}$  and  $\text{Irr}_{n-1} \phi \in \mathcal{L}_{n-1}^l$  for any  $\phi \in \mathcal{L}_n^l$ . Consequently,  $\mathcal{L}^l$  is an inductive system. The proof for  $\mathcal{R}^l$  is similar.

Lemma 2.7 (i) implies that  $\text{Irr}_{n-1} L(\omega_i^n) = \{L(\omega_{i-1}^{n-1}), L(\omega_i^{n-1})\}$ . Hence  $\mathcal{F}$  is a inductive system. For truncated symmetric powers of  $A_n(K)$  the lemma follows from Lemma 2.8.

By Lemma 2.11,  $\mathcal{L}$  is an inductive system for all types.

For the collection  $\mathcal{S}$  the result follows from Lemma 2.12 for  $G_n = B_n(K)$  or  $D_n(K)$  and Lemma 2.10 for  $G_n = C_n(K)$ . This completes the proof.  $\square$

*Proof of Theorem 1.5.* Since  $\delta(\phi) \leq \delta(\Phi)$  for all  $n$  and all  $\phi \in \Phi_n$ , by the Steinberg tensor product theorem (3) there exists an integer  $k = k(\Phi)$  such that  $\phi = \phi_0 \otimes \phi_1^{[1]} \otimes \dots \otimes \phi_k^{[k]}$  with  $\phi_j \in \text{Irr}_p G_n$ , for all  $n$  and all  $\phi \in \Phi_n$ . Fix any such  $k$ . Then the representations  $\phi_j$ ,  $0 \leq j \leq k$ , are uniquely determined (some of them can be trivial). We will use this notation until the end of the proof.

Set

$$S_n = \{\phi \in \Phi_n \mid \delta(\phi) = \delta(\Phi)\}, \quad S = \cup_{n=1}^{\infty} S_n.$$

Put

$$S_n^0 = \{\phi \in S_n \mid \delta(\phi_0) = \max_{\psi \in S} \delta(\psi_0)\}, \quad S^0 = \cup_{n=1}^{\infty} S_n^0$$

and

$$S_n^{0 \dots j} = \{\phi \in S_n^{0 \dots j-1} \mid \delta(\phi_j) = \max_{\psi \in S_n^{0 \dots j-1}} \delta(\psi_j)\}, \quad S^{0 \dots j} = \cup_{n=1}^{\infty} S_n^{0 \dots j}$$

for  $1 \leq j \leq k-1$ . Set  $T_n = S_n^{01 \dots k-1}$ .

Since  $\delta(\phi) \leq \delta(\Phi)$  for all  $\phi \in \Phi_l$  and all  $l$ , it is clear that  $T_n \neq \emptyset$  for some  $n$ . Choose minimal  $n$  with this property and denote it by  $n_{\min}$ . We claim that for each  $n \geq n_{\min}$ ,  $m > n$ , and  $\phi \in T_n$  there exists  $\psi \in T_m$  with  $\phi \in \text{Irr}_n \psi$ . Moreover, if  $\rho \in \Phi_m$  and  $\phi \in \text{Irr}_n \rho$ , then  $\rho \in T_m$ . Indeed, there exists  $\psi \in \Phi_m$  with  $\phi \in \text{Irr}_n \psi$ . Observe that  $\text{Irr}_n \psi = \cup_{(\tau^0, \dots, \tau^k)} \text{Irr}(\otimes_{j=0}^k (\tau^j)^{[j]})$ , where the union is taken over all tuples  $(\tau^0, \dots, \tau^k)$  with  $\tau^j \in \text{Irr}_n \psi_j$ . Fix a tuple  $(\tau^0, \dots, \tau^k)$  that yields  $\phi$ . Since the morphism  $\text{Fr}$  commutes with restrictions, one has  $\phi_0 = \tau_0^0$  and if  $\tau^0, \dots, \tau^{k-1} \in \text{Irr}_p G_n$ , then  $\tau^j = \phi_j$  for  $0 \leq j \leq k-1$  and  $\phi_k = \tau_0^k$ . This implies that  $\delta(\phi_0) \leq \delta(\psi_0)$  and that  $\phi_l \in \text{Irr}_n \psi_l$  for  $0 \leq l \leq k$  if  $\delta(\phi_j) = \delta(\psi_j)$  for  $0 \leq j \leq l$ . As  $\delta(\psi) \geq \delta(\phi)$  by Corollary 2.14 and  $\phi \in T_n$ , we have  $\psi \in S_m$ . Moreover, the construction of  $T_n$  yields that  $\delta(\phi_0) = \delta(\psi_0)$ . Hence  $\psi \in S_m^0$ . Using the induction on  $j$  and the argument above, we conclude that  $\delta(\phi_j) = \delta(\psi_j)$  and  $\phi_j \in \text{Irr}_n \psi_j$  for  $0 \leq k \leq j$ . Hence  $\psi \in T_m$  and the claim is proved.

For  $n \geq n_{\min}$  and  $0 \leq j \leq k$  put  $T_n^j = \{\phi_j \mid \phi \in T_n\}$ . Actually we have shown above that for each  $\rho \in T_n$  there exists  $\lambda \in T_{n+1}$  with  $\rho \in \text{Irr}_n \lambda$  and that in this situation  $\rho_j \in \text{Irr}_n \lambda_j$ . Hence the collections  $\Theta^j = \langle T_n^j \rangle$  are inductive systems. Put  $\Theta = \otimes_{j=0}^k \text{Fr}^j(\Theta^j)$  and prove that  $\Phi = \Theta$ . As  $\Phi$  is indecomposable, Lemma 3.4 implies that for every  $\phi \in \Phi_n$

there exists  $m > n$  and  $\psi \in T_m$  with  $\phi \in \text{Irr}_n \psi$ . Obviously,  $T_m \subset \Theta_m$ . Since  $\Theta$  is an inductive system, this yields that  $\Phi \subset \Theta$ . As  $\Phi$  is an inductive system, now it suffices to prove the following: if  $\rho = \otimes_{j=0}^k \rho_j^{[j]}$  with  $\rho_j \in \Theta_n^j$ , then  $\text{Irr } \rho \subset \Phi_n$ . The construction of the systems  $\Theta^j$  implies that there exist  $m > n$  and representations  $\theta_j \in T_m^j$  with  $\rho_j \in \text{Irr}_n \theta_j$ . Set  $\theta = \otimes_{j=0}^k \theta_j^{[j]}$ . As  $\Phi$  is an inductive system, now it remains to show that  $\theta \in \Phi_m$ . By the definition of  $T_m^j$ , there exist representations  $\psi^j \in T_m$  with  $\theta_j = \psi^j$ . Since  $\Phi$  is indecomposable, Corollary 3.5 implies that for some  $l > m$  there exists  $\zeta \in \Phi_l$  with  $\psi^j \in \text{Irr}_m \zeta$ . By the arguments above,  $\zeta \in T_l$  and  $\psi^j \in \text{Irr } \zeta_j$  for  $0 \leq j \leq k$ . Hence  $\theta \in \text{Irr}_m \zeta \subset \Phi_m$  as desired.  $\square$

## 4 Modules with small weight multiplicities for groups of type A

In this section  $G_n = A_n(K)$ . For a module  $M$  we assume that  $M^{\otimes 0}$  is the trivial module. Recall the pdeg function defined in (1).

**Lemma 4.1** (i) *Let  $M \in \text{Irr } G_n$  and  $\text{pdeg } M = d$ . Then  $M \in \text{Irr } V_n^{\otimes d}$ . If  $N \in \text{Irr } V_n^{\otimes d}$ , then  $\text{pdeg } N \leq d$ .*

(ii) *Let  $M \in \text{Irr } G_n$  and  $\text{pdeg } M^* = d$ . Then  $M \in \text{Irr}(V_n^*)^{\otimes d}$ . If  $N \in \text{Irr}(V_n^*)^{\otimes d}$ , then  $\text{pdeg } N^* \leq d$ .*

*Proof.* (i) Let  $M_{\mathbb{C}}$  be the irreducible module for the group  $A_n(\mathbb{C})$  with the same highest weight as  $M$ . Denote by  $V_{\mathbb{C}}$  the natural  $A_n(\mathbb{C})$ -module. By [10],  $M_{\mathbb{C}} \in \text{Irr } V_{\mathbb{C}}^{\otimes d}$ . Obviously, the formal characters of the modules  $V_{\mathbb{C}}^{\otimes d}$  and  $V^{\otimes d}$  coincide. By [11], these modules have filtrations by Weyl modules. Denote by  $m(\omega)$  and  $m_{\mathbb{C}}(\omega)$  the multiplicities of the modules  $V(\omega)$  in these filtrations for  $V^{\otimes d}$  and  $V_{\mathbb{C}}^{\otimes d}$ , respectively. Then  $m(\omega) = m_{\mathbb{C}}(\omega)$  as the weight multiplicities in  $V^{\otimes d}$  and  $V_{\mathbb{C}}^{\otimes d}$  coincide. Since  $M \in \text{Irr } V(\omega(M))$ , this forces  $M \in \text{Irr } V^{\otimes d}$ . The first claim of part (i) is proved.

Recall that  $\omega_i^n = \varepsilon_1 + \dots + \varepsilon_n$ ,  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  for  $1 \leq i \leq n$ , and  $\varepsilon_1 + \dots + \varepsilon_{n+1} = 0$ . This implies that if  $\text{pdeg } N = k$  and  $\omega(M) = \sum_{i=1}^n b_i \varepsilon_i$ , then  $\sum_{i=1}^n b_i = k$ . It is clear that each weight  $\mu \in \Lambda(V_n^{\otimes d})$  has the form  $d\omega_1 - \sum_{i=1}^n c_i \alpha_i$  with  $c_i \in \mathbb{Z}_{\geq 0}$ . This yields that  $\text{pdeg } N \leq d$  for  $N \in \text{Irr } V_n^{\otimes d}$  and completes the proof of (i).

(ii) Take into account that  $(V_n^*)^{\otimes d} \cong (V_n^{\otimes d})^*$ .  $\square$

Recall the sets  $\mathcal{L}_n^d$  and  $\mathcal{R}_n^d$  defined in (2).

**Proposition 4.2**  $\mathcal{L}_n^d = \{M \in \text{Irr } G_n \mid \text{pdeg } M \leq d\}$ ,  $\mathcal{R}_n^d = \{M \in \text{Irr } G_n \mid \text{pdeg } M^* \leq d\}$ .

*Proof.* This follows immediately from Lemma 4.1.  $\square$

**Proposition 4.3** *Let  $M \in \text{Irr}_p G_n$  and  $\omega(M) \notin \Omega_p(G_n)$ . Assume that  $\text{pdeg } M \leq n$ . Then  $\text{wdeg } M \geq \text{pdeg } M - 2$ .*

*Proof.* Put  $d = \text{pdeg } M$  and  $H = G(1, \dots, d-1)$ . Then  $H \cong SL_d(K)$ . Recall that  $\omega(M) \notin \Omega_p(G_n)$ , hence  $d > 1$ . Denote by  $N$  the Smith factor of  $M$  associated with  $H$  (see 2.3). Since  $\omega(M)$  is not fundamental, one easily observes that  $\omega(M) = \sum_{i=1}^{d-1} a_i \omega_i$  and hence  $\text{pdeg } M = \text{pdeg } N = d$ .

Now we can apply the Schur functor to the  $H$ -module  $N$ . Let  $M(d, d)$  be the category of the polynomial  $GL_d(K)$ -modules over  $K$  which are homogeneous of degree  $d$ ,  $\Sigma_d$  be the symmetric group of degree  $d$ , and let  $K\Sigma_d - \text{mod}$  be the category of  $K\Sigma_d$ -modules. Let  $V \in M(d, d)$  and  $\lambda = b_1\varepsilon_1 + \dots + b_d\varepsilon_d$  be the highest weight of  $V$ . Here  $\varepsilon_i$  are weights of  $GL_d(K)$ . Note that  $b_1 \geq \dots \geq b_d \geq 0$  and  $b_1 + \dots + b_d = d$ . Hence  $\lambda = (b_1, \dots, b_d)$  is a partition of  $d$ .

The Schur functor

$$\mathcal{S}_d : M(d, d) \rightarrow K\Sigma_d - \text{mod}$$

is defined as  $\mathcal{S}_d(V) = V^0$  where  $V^0$  is the  $(1, \dots, 1)$ -weight subspace of the module  $V$  [12, Chapter 6]. Alternatively, one can regard  $V$  as an  $SL_d(K)$ -module and define  $\mathcal{S}_d(V)$  as the 0-weight subspace of  $V$ .

The functor  $\mathcal{S}_d$  is exact and by [12, 6.4],

$$\mathcal{S}_d(N) \cong D^{\lambda'} \otimes \text{sgn}$$

where  $D^{\lambda'}$  is the irreducible  $\Sigma_d$ -module corresponding to the partition  $\lambda'$  dual to  $\lambda$ , and  $\text{sgn}$  is the sign module for  $\Sigma_d$ . Hence by Proposition 2.15,  $\text{wdeg } M \geq \text{wdeg } N \geq \dim D^{\lambda'}$ . If  $\text{wdeg } N < d - 2$ , then [13] implies that  $D^{\lambda'} \otimes \text{sgn}$  is equal to the trivial module or  $\text{sgn}$ . So  $D^{\lambda'}$  is the trivial module or  $\text{sgn}$  in this case. If  $D^{\lambda'}$  is trivial, then its diagram is the row of  $d$  boxes, therefore the diagram for  $\lambda$  is the column of  $d$  boxes and  $N$  and  $M$  are fundamental modules (recall that their highest weights are determined by the same formula). By [16, Section 5], if  $d = k(p - 1) + r$  with  $0 \leq r < p - 1$ , then the diagram for  $\text{sgn}$  consists of  $r$  rows of length  $k + 1$  and  $p - 1 - r$  rows of length  $k$ . In this case  $\lambda$  has the diagram of  $k$  rows of length  $p - 1$  and 1 row of length  $r$  and so  $\omega(N) = (p - 1 - r)\omega_k + r\omega_{k+1}$  which implies that  $N$  and  $M$  are truncated symmetric powers of the natural modules. In both cases  $\omega(M) \in \Omega_p(G_n)$  which yields a contradiction. Hence  $\text{wdeg } M \geq \text{wdeg } N \geq d - 2$ .  $\square$

**Lemma 4.4** *Let  $n \geq d$ . Then  $\text{wdeg } V_n^{\otimes d} = d!$ .*

*Proof.* Set  $T = V_n^{\otimes d}$ . Note that each weight  $\lambda$  of  $T$  is of the shape  $\lambda = b_1\varepsilon_1 + \dots + b_d\varepsilon_d$  where  $(b_1, \dots, b_d)$  runs over all  $b_i \geq 0$  with  $b_1 + \dots + b_d = d$  and  $\dim T^\lambda = \frac{d!}{b_1!b_2!\dots b_d!} \leq d!$ . On the other hand, for  $\lambda = \varepsilon_1 + \dots + \varepsilon_d$ , this dimension is exactly  $d!$ . Therefore,  $\text{wdeg } V_n^{\otimes d} = d!$ .  $\square$

**Proposition 4.5** *Let  $n \geq d$  and  $M = L(a_1\omega_1 + \dots + a_d\omega_d) \in \mathcal{L}_n^d$  or  $M = L(a_d\omega_{n-d+1} + \dots + a_1\omega_n) \in \mathcal{R}_n^d$ . Then  $\text{wdeg } M \leq d!$ . Moreover,  $\text{wdeg } M$  is determined by the sequence  $(a_1, \dots, a_d)$  and does not depend on  $n$ .*

*Proof.* Let  $M \in \mathcal{L}_n^d$ . We have  $\text{pdeg } M = j \leq d$  by Lemma 4.1(i). Set  $T = V_n^{\otimes j}$ . Observe that  $M \in \text{Irr } T$  by the same lemma. Therefore  $\text{wdeg } M \leq j! \leq d!$  by Lemma 4.4.

Let  $\lambda \in \Lambda(M)$  be dominant. As  $\lambda \in \Lambda(T)$ , we have  $\lambda = b_1\varepsilon_1 + \dots + b_j\varepsilon_j$  with  $b_1 \geq \dots \geq b_j \geq 0$ ,  $b_i \in \mathbb{Z}_{\geq 0}$  and  $b_1 + \dots + b_j = j$ . Set  $\omega = \omega(M)$ . Then  $\lambda = j\omega_1 - \sum_{i=1}^{j-1} c_i\alpha_i = \omega - \sum_{i=1}^{j-1} d_i\alpha_i$  with  $c_i, d_i \in \mathbb{Z}_{\geq 0}$ . Denote by  $M_S$  the Smith factor of  $M$  associated with the subgroup  $G(1, \dots, j-1) \cong G_{j-1}$ . By Theorem 2.3,  $\dim M^\lambda = \dim M_S^{\lambda_S}$  for the weight  $\lambda_S = \lambda \downarrow G(1, \dots, j-1)$ . Since each weight in  $\Lambda(M)$  lies in the same orbit with a dominant weight under the action of the Weyl group, we conclude that  $\text{wdeg } M = \text{wdeg } M_S$  and hence does not depend on  $n$ . To handle the case  $M \in \mathcal{R}_n^d$ , consider  $M^*$ .  $\square$

**Lemma 4.6** *Let  $1 \leq j < k \leq n$ , and let  $\omega = \sum_{s=j}^k a_s \omega_s^n$  be a dominant  $p$ -restricted weight of  $G_n$  with both  $a_j$  and  $a_k \neq 0$ . Then*

$$\text{wdeg } L(\omega) \geq k - j.$$

*Proof.* Write  $\omega = a_j \omega_j + a_{i_1} \omega_{i_1} + \dots + a_{i_t} \omega_{i_t} + a_k \omega_k$  with  $j < i_1 < \dots < i_t < k$  and  $a_{i_1}, \dots, a_{i_t} \neq 0$  ( $t$  can be zero). By [17, Proposition 1.21],  $\text{wdeg } L(\omega) \geq f(j, i_1, \dots, i_t, k)$ , where for  $l$ -tuples  $(u_1, \dots, u_l)$  with  $u_1 < \dots < u_l$  the integers  $f(u_1, \dots, u_l)$  are determined by the following recurrent relations:

$$f(u_1) = 1;$$

$$f(u_1, u_2) = u_2 - u_1;$$

$$f(u_1, u_2, \dots, u_l) = (u_2 - u_1)f(u_2, \dots, u_l) + f(u_3, \dots, u_l) \quad \text{for } l > 2.$$

We claim that  $f(j, i_1, \dots, i_t, k) \geq k - j$ . For  $t = 0$  this holds by definition. Then apply induction on  $t$ . Let  $t > 0$ . One easily concludes that  $f(u_1, \dots, u_l) \geq 1$  for all positive integers  $u_1, \dots, u_l$ . Now the induction hypothesis yields that

$$f(j, i_1, \dots, i_t, k) = (i_1 - j)f(i_1, \dots, i_t, k) + f(i_2, \dots, i_t, k) \geq (i_1 - j)(k - i_1) + 1.$$

(For  $t = 1$  we have  $f(j, i_1, k) = (i_1 - j)(k - i_1) + 1$ .) Note that  $ab \geq a + b$  for  $a$  and  $b \in \mathbb{N}$  and  $a, b > 1$ . Hence  $ab + 1 \geq a + b$  for all  $a$  and  $b \in \mathbb{N}$ . This yields our claim and completes the proof.  $\square$

Propositions 4.3 and 4.5 imply that for groups of type  $A_n$  there exist classes of simple modules  $M$  with  $\text{wdeg } M$  arbitrary large, but small with respect to  $n$ . Note that for a generic simple  $p$ -restricted module  $\text{wdeg } M$  grows with the growth of  $n$ .

**Proposition 4.7** *Let  $M \in \text{Irr}_p G_n$ ,  $\omega(M) \notin \Omega_p(G_n)$ , and  $n \geq 16$ . Assume  $\text{pdeg } M > n$  and  $\text{pdeg } M^* > n$ . Then  $\text{wdeg } M > \sqrt{n}/p - 1$ .*

*Proof.* Let  $\omega = \sum_{t=i}^j a_t \omega_t$  with  $a_i a_j \neq 0$ ,  $1 \leq i \leq j \leq n$ . Due to Lemma 4.6 one can assume that  $j - i \leq \sqrt{n}/p - 1$  (otherwise  $\text{wdeg } L(\omega) \geq j - i > \sqrt{n}/p - 1$  as required). Put  $k = j - i + 1$  and  $a = \sum_{t=i}^j a_t$ . Then  $k \leq \sqrt{n}/p$  and

$$a \leq k(p - 1) < \sqrt{n}. \tag{12}$$

Passing to  $M^*$  if necessary, one can assume  $i - 1 \leq n - j$ . Denote by  $H_s$  ( $s = 1, 2, \dots, i$ ) the subgroup  $G(s, \dots, n) \cong A_{n-s+1}(K)$ . So  $H_1 = G$  and the rank of  $H_s$  is equal to  $n - s + 1 > n/2$  for all  $s \leq i$ .

Let  $L_s$  be the Smith factor of  $L(\omega)$  with respect to  $H_s$ . Then  $\text{pdeg } L_s = \text{pdeg } L_i + (i - s)a$ . Note that

$$\text{pdeg } L_i \leq ka \leq k^2(p - 1) \leq n(p - 1)/p^2 < n/2$$

since  $p \geq 2$ .

Fix minimal  $s$  such that  $\text{pdeg } L_s \leq n/2$ . Since  $\text{pdeg } L_1 = \text{pdeg } L(\omega) > n$ , we have  $s > 1$ . Then  $\text{pdeg } L_{s-1} = \text{pdeg } L_s + a > n/2$ , so  $\text{pdeg } L_s > n/2 - a$ . Applying (12), we get  $n/2 - a > n/2 - \sqrt{n}$ . As the rank of  $H_s$  is greater than  $n/2$ , by Lemma 4.3,

$$\text{wdeg } L(\omega) \geq \text{wdeg } L_s \geq \text{pdeg } L_s - 2 > n/2 - \sqrt{n} - 2 = \sqrt{n}(\sqrt{n}/2 - 1) - 2 \geq \sqrt{n} - 2 > \sqrt{n}/p - 1$$

since  $n \geq 16$  and  $p \geq 2$ .  $\square$

Now we are ready to prove our first main result.

*Proof of Theorem 1.6.* Part (i) is proved in Proposition 4.7 and part (ii) is proved in Propositions 4.3 and 4.5.  $\square$

**Corollary 4.8** *Let  $M \cong \otimes_{k=0}^i M_k^{[k]}$ . If at least one of  $M_k$  satisfies the assumptions of Proposition 4.7, then  $\text{wdeg } M > \sqrt{n}/p - 1$ .*

*Proof.* This follows immediately from Lemma 2.16 and Proposition 4.7.  $\square$

Now we pass to modules that are not  $p$ -restricted.

**Lemma 4.9** *Let  $M \in \text{Irr } G_n$ ,  $M = N_1 \otimes N_2^{[s]}$ ,  $N_1, N_2 \in \text{Irr } G_n$ , and let  $\delta(N_1) < p^s$ . Then for any weight  $\lambda \in \Lambda(M)$  there exists a unique pair  $(\mu, \nu)$  with  $\mu \in \Lambda(N_1)$ ,  $\nu \in \Lambda(N_2^{[s]})$ , and  $\lambda = \mu + \nu$ .*

*Proof.* It is obvious that  $\lambda = \mu + \nu$  for some  $\mu$  and  $\nu$ . Put  $N' = N_2^{[s]}$ . Suppose that  $\mu + \nu = \mu' + \nu'$  with  $\mu' \in \Lambda(N_1)$ ,  $\nu' \in \Lambda(N')$ , and  $\mu \neq \mu'$ . Then  $\mu - \mu' = \nu' - \nu$ . Acting by the Weyl group, one can assume that  $\mu - \mu'$  (and hence  $\nu' - \nu$ ) is dominant. Denote by  $\alpha_m$  the maximal root of  $G_n$ . Note that  $\nu = p^s \xi$  and  $\nu' = p^s \xi'$  with  $\xi$  and  $\xi' \in \Lambda(N_2)$ . Therefore

$$p^s \langle \xi' - \xi, \alpha_m \rangle = \langle \nu' - \nu, \alpha_m \rangle = \langle \mu - \mu', \alpha_m \rangle \leq 2\delta(N_1) < 2p^s.$$

This implies that  $\langle \xi' - \xi, \alpha_m \rangle = 1$ , i.e.  $\xi' - \xi$  is a fundamental weight. However, this difference is a radical weight (i.e. a linear combination of roots). This yields a contradiction and proves the lemma.  $\square$

Now consider tensor products of certain special modules with relatively small  $\text{wdeg } M$ .

**Theorem 4.10** *Let  $d \in \mathbb{N}$  and*

$$M = N_0 \otimes \dots \otimes N_l \in \text{Irr } G_n. \quad (13)$$

*Assume that*

$$N_t = \otimes_{s=i_{t-1}+1}^{i_t} M_s^{[s]} \quad (14)$$

*with  $i_{-1} = -1$  and for each  $t$ ,  $0 \leq t \leq l$ , one of the following holds:  $M_s \in \mathcal{L}_n^d$  for  $i_{t-1} + 1 \leq s \leq i_t$ , or  $M_s \in \mathcal{R}_n^d$  for all these  $s$ , or  $\omega(N_t) \in \Omega(G_n)$ . Let  $\delta(N_f) < p^{i_f+1}$  for all  $N_f$  with  $f < l$  ( $i_f$  are such as in (14)). Then  $\text{wdeg } M < m$  for some integer  $m$  that depends only on  $d$  and the differences  $i_t - i_{t-1}$ , and does not depend on  $n$ .*

*Proof.* If  $\omega(M) \in \Omega(G_n)$ , we have  $\text{wdeg } M = 1$  by [28, Theorem 2]. Assume that this is not the case. Let  $S = \{u_1, \dots, u_k\}$  be the set of all indices  $t$  for which  $\omega(N_t) \notin \Omega(G_n)$ . Set  $s_j = i_{u_j-1} + 1$  and  $l_j = i_{u_j} - s_j$  for  $1 \leq j \leq k$ . Put  $N'_j = \otimes_{g=s_j}^{l_j} M_{s_j+g}^{[g]}$ . We have  $N_{u_j} = (N'_j)^{[s_j]}$ . Hence  $\text{wdeg } N_{u_j} = \text{wdeg } N'_j$ . Put  $d_j = d(1 + p + \dots + p^{l_j})$ . We claim that  $\text{wdeg } M \leq \prod_{j=1}^k d_j!$ .

Apply induction on  $l$ . For  $l = 0$  one has  $M \in \mathcal{L}^{d_0}$  or  $\mathcal{R}^{d_0}$ . Hence our assertion follows from Proposition 4.5. Assume that  $l > 0$  and the assertion holds for  $l - 1$ . Set  $M' = N_0 \otimes \dots \otimes N_{l-1}$ . Since  $\delta(N_j) < p^{i_j+1}$  for  $j < l$ , we get  $\delta(M') < p^{i_{l-1}+1}$ . Then by Lemma 4.9, for each  $\lambda \in \Lambda(M)$  there exist a unique pair  $(\mu, \nu)$  with  $\mu \in \Lambda(M')$ ,

$\nu \in \Lambda(N_l)$ , and  $\lambda = \mu + \nu$ . Then  $\dim M^\lambda = \dim((M')^\mu) \dim(N_l^\nu)$  and hence  $\text{wdeg } M = \text{wdeg } M' \text{ wdeg } N_l$ . By the induction assumptions,  $\text{wdeg } M' \leq \prod_{j=1}^{k-1} d_j!$  if  $u_k = l$  and  $\text{wdeg } M' \leq \prod_{j=1}^k d_j!$  otherwise. In the first case  $\text{wdeg } N_l \leq d_k!$  by Proposition 4.5. In the second one  $\omega(N_l) \in \Omega(G_n)$  and  $\text{wdeg } N_l = 1$ . Another application of Lemma 4.9 completes the proof.  $\square$

**Remark** In some cases much stronger estimates can be obtained. In particular, this holds if  $M = \otimes_{k=0}^f M_k^{[k]}$  with  $M_k \in \text{Irr}_p G_n$  and  $\delta(M_k) < p$  for all  $k < f$ . Then, applying Lemma 4.9 and Proposition 4.5, we can deduce that  $\text{wdeg } M \leq (d!)^N$ , where  $N$  is the number of indices  $k$  for which  $\omega(M_k) \notin \Omega_p(G_n)$ .

Proposition 4.11 shows that our assumptions on  $\delta(N_f)$  play a crucial role in Theorem 4.10.

**Proposition 4.11** *Let  $i, l \in \mathbb{N}$  with  $i < l$  and  $M, N \in \text{Irr } G_n$ . Assume that  $\omega(M) = \sum_{t=1}^i a_t \omega_t = \sum_{k=0}^j p^k \lambda_k$  with  $\lambda_k$   $p$ -restricted and  $\omega(N) = \sum_{t=l}^n b_t \omega_t$ . Suppose that  $\delta(M) \geq p^{j+1}$ . Set  $Q = M \otimes N^{[j+1]}$ . Then  $\text{wdeg } Q \geq l - i - 1$ . The same holds if  $\omega(M) = \sum_{t=l}^n b_t \omega_t$ ,  $\omega(N) = \sum_{t=1}^i a_t \omega_t$  and other assumptions of the proposition hold. In particular, in this situation  $\text{wdeg } Q \geq n - m - i$  if  $M \in \mathcal{L}_n^i$ ,  $N \in \mathcal{R}_n^m$  or vice versa.*

*Proof.* We will consider the case where  $\omega(M) = \sum_{t=1}^i a_t \omega_t$  and  $\omega(N) = \sum_{t=l}^n b_t \omega_t$ . The proof for the other case is similar.

Taking minimal possible  $l$ , we can suppose that  $b_l \neq 0$ . Put  $c = \delta(M)$  and write down the  $p$ -adic expansion  $c = \sum_{k=0}^u c_k p^k$  with  $0 \leq c_k < p$ .

(a) First assume that  $c_{j+1} \neq 0$ . Set  $\Gamma = G(\alpha_1 + \dots + \alpha_i, \alpha_{i+1}, \dots, \alpha_n)$ . Observe that  $\Gamma$  is conjugate to  $G_{n-i+1}$ , the group  $G(i+1, \dots, n)$  is conjugate to  $G_{n-i}$  and  $G(i+1, \dots, l)$  is conjugate to  $G_{l-i}$ . Then one easily concludes that  $L(c\omega_1) \in \text{Irr}_{n-i+1} M$ . By the Steinberg tensor product theorem (3),  $L(c\omega_1) = \otimes_{k=0}^u L(c_k \omega_1)^{[k]}$ . Let  $0 < a < p$  and  $a\omega_1 \in \Lambda(G_{n-i+1})$ . Applying [29, Proposition 1.4], one obtains that  $L(\omega_1)$  and  $L(0) \in \text{Irr}_{n-i} L(a\omega_1)$ . Hence  $L(p^{j+1}\omega_1) \in \text{Irr}_{n-i}(L(c\omega_1)) \subset \text{Irr}_{n-i} M$ . The arguments above imply that  $L(p^{j+1}\omega_1) \in \text{Irr}_{l-i} M$ . Analyzing the restriction  $M \downarrow G(i+1, \dots, l)$ , we get that  $L(b_l p^{j+1}\omega_{l-i}) \in \text{Irr}_{l-i} N$ . Consequently,  $L(p^{j+1}(\omega_1 + b_l \omega_{l-i})) \in \text{Irr}_{l-i}(M \otimes N)$ . Lemma 4.6 implies  $\text{wdeg}(M \otimes N) \geq l - i - 1$ .

(b) Now let  $c_{j+1} = 0$ . Then  $\sum_{k=1}^i a_k = c \geq p^{j+2}$ . Fix minimal  $s$  with  $\sum_{k=1}^s a_k \geq \sum_{k=0}^j c_k p^k$ . Since all  $a_k < p^{j+1}$ , we get  $s < i$ . Identify the group  $G_{n-s}$  with  $G(s+1, \dots, n)$ . Let  $M_s$  be the Smith factor of  $M$  with respect to  $G_{n-s}$ . Denote by  $c_s$  the value of  $\omega(M_s)$  on the maximal root of  $G_{n-s}$  and put  $\Sigma_s = a_1 + \dots + a_s$ . One has  $c_s = c - \Sigma_s$ . Write  $c_s = \sum_{k=0}^f c_k^s p^k$  with  $0 \leq c_k^s < p$ . One can observe that  $c_{j+1}^s = p - 1$ . Now we can proceed as in part (a) using the group  $G_{n-s}$  rather than  $G_{n-i}$ .  $\square$

## 5 Inductive systems with bounded weight multiplicities

In this section we classify all BWM-systems for all four classical series of algebraic groups. We will denote by  $\mathbb{N}_j$  the set of integers  $s$  with  $0 \leq s \leq j$ . Recall the systems  $C_L(a_1, \dots, a_d)$  and  $C_R(a_1, \dots, a_d)$  defined in (7) and (8).

**Lemma 5.1** *Assume that  $k_1 < \dots < k_t \in \mathbb{Z}_{\geq 0}$  and  $a_{ij} < p$  for  $1 \leq i \leq d$ ,  $0 \leq j \leq t$ . Set  $a_i = \sum_{j=0}^t a_{ij} p^{k_j}$ ,  $1 \leq i \leq d$ . Then*

$$C_L(a_1, \dots, a_d) = \otimes_{j=0}^t C_L(a_{1j}, \dots, a_{dj})^{[k_j]}.$$



The same holds for  $C_R(a_1, \dots, a_d)$ .

*Proof.* Put  $C_L(a_1, \dots, a_d) = L$ ,  $C_L(a_{1j}, \dots, a_{dj}) = L^j$ , and  $\otimes_{j=0}^t (L^j)^{[k_j]} = L'$ . For  $n \geq d$  set  $M_n = L(a_1\omega_1^n + \dots + a_d\omega_d^n) \in \text{Irr } G_n$  and  $M_n^j = L(a_{1j}\omega_1^n + \dots + a_{dj}\omega_d^n) \in \text{Irr}_p G_n$ . Then  $M_n = \otimes_{j=0}^t (M_n^j)^{[k_j]}$  by the Steinberg tensor product theorem (3). Now Theorem 2.3 implies that  $M_n \in L'_n$  and hence  $L \subset L'$ . Taking into account the definition of a tensor product of inductive systems, it remains to prove that for each collection  $(N_1, \dots, N_t)$  with  $N_j \in L_n^j$  the set  $S = \text{Irr}(\otimes_{j=0}^t N_j^{[k_j]}) \subset L_n$ . The construction of the systems  $L^j$  implies that for  $q$  large enough  $L_n^j \subset \text{Irr}_n M_q^j$  for all  $j$ ,  $0 \leq j \leq t$ . Hence  $S \subset \text{Irr}_n M_q \subset L_n$ . This completes the proof for  $C_L(a_1, \dots, a_d)$ . The proof for  $C_R(a_1, \dots, a_d)$  is similar.  $\square$

Now we start describing BWM-systems for groups of type  $A_n$ . Note that  $\mathcal{F} = \mathcal{T}$  for  $p = 2$ , but this does not affect the proofs.

**Proposition 5.2** *Assume that  $S_1 \cup S_2 \cup S_3 = \mathbb{N}_j$ ,  $S_i \cap S_k = \emptyset$  for  $i \neq k$ ,  $S_3 = \emptyset$  if  $p = 2$ , and  $S_2 \cup S_3 \neq \emptyset$ . If  $S_1 \neq \emptyset$ , for each  $k \in S_1$  set  $M_{n,k} = M_{n,L}(a_{1k}, \dots, a_{dk})$  or  $M_{n,R}(a_{1k}, \dots, a_{dk})$ , where  $0 \leq a_{1k}, \dots, a_{dk} < p$  and the index “L” or “R” is the same for all  $n$ . Put  $\Psi^k = \langle M_{n,k} \mid n \geq d \rangle$  for  $k \in S_1$ ,  $\Psi^k = \mathcal{F}$  for  $k \in S_2$ ,  $\Psi^k = \mathcal{T}$  for  $k \in S_3$ , and  $\Psi = \otimes_{k=0}^j \text{Fr}^k(\Psi^k)$ . Let  $\Phi$  be an inductive system. Assume that for each  $l$  there exist  $n$  and a module  $\phi = \otimes_{k=0}^j \phi_k^{[k]} \in \Phi_n$  with the following properties:*

$$\phi_k = M_{n,k} \text{ for } k \in S_1; \quad (15)$$

$$\phi_k \in \mathcal{F}_n \text{ for } k \in S_2; \quad (16)$$

$$\phi_k \in \mathcal{T}_n \text{ for } k \in S_3; \quad (17)$$

$$\phi_k \notin \mathcal{L}_n^l \cup \mathcal{R}_n^l \text{ for } k \in S_2 \cup S_3. \quad (18)$$

Then  $\Psi \subset \Phi$ .

*Proof.* The construction of  $\Psi$  and the definition of the tensor product of inductive systems imply that it suffices to prove that for each  $m \geq d$  and each  $G_m$ -module  $\pi = \otimes_{k=0}^j \pi_k^{[k]}$  with  $\pi_k = M_{m,k}$  for  $k \in S_1$ ,  $\pi_k \in \mathcal{F}_m$  for  $k \in S_2$ , and  $\pi_k \in \mathcal{T}_m$  for  $k \in S_3$ , one has  $\pi \in \Phi_m$ . Put  $l = (p-1)(m+1)$  and choose  $n > m$  and  $\phi \in \Phi_n$  that satisfies (15)-(18) for this  $l$ . Then Lemma 2.7 and Corollary 2.9 imply that  $\pi_k \in \text{Irr}_m \phi_k$  for  $k \in S_2 \cup S_3$ . By Theorem 2.3,  $\pi_k \in \text{Irr}_m \phi_k$  for  $k \in S_1$ . Hence  $\pi \in \text{Irr}_m \phi \subset \Phi_m$ . This completes the proof.  $\square$

Note that  $S_1$  can be empty.

**Corollary 5.3** *Set  $n' = \lfloor \frac{n+1}{2} \rfloor$  and  $F_n = L(\omega_{n'}^n) \in \text{Irr } G_n$ . Then  $\mathcal{F} = \langle F_n \mid n \in \mathbb{N} \rangle$ .*

*Proof.* Theorem 2.3 implies that  $F_n \in \text{Irr}_n F_{n+1}$ . Hence  $\langle F_n \mid n \in \mathbb{N} \rangle$  is an inductive system. Naturally, for each  $d$  there exists  $n$  with  $F_n \notin \mathcal{L}_n^d \cup \mathcal{R}_n^d$ . Now apply Proposition 5.2.  $\square$

**Corollary 5.4** *Define  $n'$  as in Corollary 5.3 and set  $T_n = L((p-1)\omega_{n'})$ . Then  $\mathcal{T} = \langle T_n \mid n \in \mathbb{N} \rangle$ .*

*Proof.* Argue as in the proof of Corollary 5.3, applying Propositions 4.2 and 5.2.  $\square$

**Corollary 5.5** *Let  $\Phi \subset \mathcal{F}$  or  $\mathcal{T}$  be a proper inductive subsystem. Then  $\Phi \subset \mathcal{L}^d \cup \mathcal{R}^d$  for some  $d \in \mathbb{N}$ .*

**Proposition 5.6** *Let  $\Phi \subset \mathcal{L}^d$  or  $\mathcal{R}^d$ . Then  $\Phi$  is a finite union of systems  $C_L(a_1, \dots, a_d)$  or  $C_R(a_1, \dots, a_d)$ , respectively.*

*Proof.* It is enough to prove for  $\mathcal{L}^d$ . The proof for  $\mathcal{R}^d$  is similar. Assume that  $\Phi \subset \mathcal{L}^d$ . For a  $d$ -tuple  $s = (a_1, \dots, a_d)$  with  $a_j \in \mathbb{Z}_{\geq 0}$  set  $M_{n,L}(s) = L(a_1\omega_1^n + \dots + a_d\omega_d^n)$  and  $C_L(s) = C_L(a_1, \dots, a_d)$ . Denote by  $S_d$  the set of all such tuples with  $a_1 + 2a_2 + \dots + da_d \leq d$ . Obviously, the set  $S_d$  is finite. By Proposition 4.2,  $\mathcal{L}_n^d = \{M_{n,L}(s) \mid s \in S_d\}$ . Let  $S(\Phi) = \{s \in S_d \mid C_L(s) \subset \Phi\}$  and  $\Psi = \cup_{s \in S(\Phi)} C_L(s)$ . We claim that  $\Psi = \Phi$ . Suppose this is not the case and set  $D_n = \Phi_n \setminus \Psi_n$ . Then  $D_n \neq \emptyset$  for large enough  $n$ . Hence, there exists  $\sigma \in S_d$  for which the set  $\{n \mid M_{n,L}(\sigma) \in D_n\}$  is infinite. Since  $\Phi$  and  $\Psi$  are inductive systems, this implies that  $C_L(\sigma) \subset \Phi$  and yields a contradiction. Hence  $\Psi = \Phi$  as desired.  $\square$

**Corollary 5.7** *If  $\Phi \subset \mathcal{L}^d$  or  $\mathcal{R}^d$  is an indecomposable inductive system, then  $\Phi = C_L(a_1, \dots, a_d)$  or  $C_R(a_1, \dots, a_d)$ , respectively.*

**Lemma 5.8** *Let  $\Phi \subset \mathcal{L}^a \cup \mathcal{R}^b$ , but  $\Phi \not\subset \mathcal{L}^a$  and  $\Phi \not\subset \mathcal{R}^b$ . Then  $\Phi = \Phi^L \cup \Phi^R$  where  $\Phi^L$  and  $\Phi^R$  are proper subsystems of  $\Phi$ ,  $\Phi^L \subset \mathcal{L}^a$ , and  $\Phi^R \subset \mathcal{R}^b$ .*

*Proof.* Set  $\Pi_n = \Phi_n \cap \mathcal{L}_n^a$ ,  $\Sigma_n = \Phi_n \cap \mathcal{R}_n^b$ . Observe that  $\Pi_n \cap \Sigma_n = \emptyset$  for  $n \geq a + b$ . As  $\Phi$ ,  $\mathcal{L}^a$ , and  $\mathcal{R}^b$  are inductive systems, this implies that  $\Phi^L = \langle \Pi_n \mid n \geq a + b \rangle$  and  $\Phi^R = \langle \Sigma_n \mid n \geq a + b \rangle$  are inductive systems. It is clear that  $\Phi_n = \Phi_n^L \cup \Phi_n^R$ . Hence  $\Phi = \Phi^L \cup \Phi^R$ .  $\square$

**Proposition 5.9** *Let  $\Phi$  be a  $p$ -restrictedly generated BWM-system. Then one of the following holds:*

- (1)  $\Phi = \mathcal{F}$ ;
- (2)  $\Phi = \mathcal{T}$ ;
- (3)  $\Phi = \mathcal{F} \cup \mathcal{T}$ ;
- (4)  $\Phi \subset \mathcal{L}^d \cup \mathcal{R}^d$ ;
- (5)  $\Phi = \Phi' \cup \mathcal{T}$ ,  $\Phi = \Phi' \cup \mathcal{F}$  or  $\Phi = \Phi' \cup \mathcal{F} \cup \mathcal{T}$  with  $\Phi' \subset \mathcal{L}^d \cup \mathcal{R}^d$ .

*In all cases, if  $\text{wdeg } \Phi = k$ , then  $\Phi \subset \mathcal{L}^{k+2} \cup \mathcal{R}^{k+2} \cup \mathcal{F} \cup \mathcal{T}$ .*

*Proof.* Assume that  $\text{wdeg } \Phi = k$ . First suppose that  $\Phi \not\subset \mathcal{F} \cup \mathcal{T}$ . Then  $\Phi_n \not\subset \mathcal{F}_n \cup \mathcal{T}_n$  for large enough  $n$ . Fix  $n > (k+1)^2 p^2$  and a  $p$ -restricted  $\phi \in \Phi_n \setminus \{\mathcal{F}_n \cup \mathcal{T}_n\}$ . Proposition 4.7 implies that  $\text{pdeg } \phi$  or  $\text{pdeg } \phi^* \leq n$  since otherwise  $\text{wdeg } \phi > \sqrt{n}/p - 1 > k$ . Now Proposition 4.3 forces that  $\text{pdeg } \phi$  or  $\text{pdeg } \phi^* \leq k + 2$  and hence  $\phi \in \mathcal{L}_n^{k+2}$  or  $\mathcal{R}_n^{k+2}$  by Proposition 4.2. This yields the last claim of the proposition.

Now we want to reduce the problem to the case where both  $\mathcal{F} \not\subset \Phi$  and  $\mathcal{T} \not\subset \Phi$ . Assume that this is not the case. Put  $\Psi = \mathcal{F}$  if  $\mathcal{F} \subset \Phi$ , but  $\mathcal{T} \not\subset \Phi$ ;  $\Psi = \mathcal{T}$  if  $\mathcal{T} \subset \Phi$ , but  $\mathcal{F} \not\subset \Phi$ ; and  $\Psi = \mathcal{F} \cup \mathcal{T}$  if  $\mathcal{F} \cup \mathcal{T} \subset \Phi$ . If  $\Psi = \Phi$ , the proposition is proved. Assume  $\Psi \neq \Phi$  and put  $D = D(\Phi, \Psi)$ . We claim that both  $\mathcal{F} \not\subset D$  and  $\mathcal{T} \not\subset D$ .

If  $\Psi \neq \mathcal{T} \cup \mathcal{F}$ , define an inductive system  $D'$  by the equality  $\{\Psi, D'\} = \{\mathcal{F}, \mathcal{T}\}$ . Then  $D = \langle \Delta_n \mid n \in \mathbb{N} \rangle$ , where  $\Delta_n \subset \mathcal{L}_n^{k+2} \cup \mathcal{R}_n^{k+2}$  or  $\mathcal{L}_n^{k+2} \cup \mathcal{R}_n^{k+2} \cup D'_n$ . Hence  $D \subset \mathcal{L}^{k+2} \cup \mathcal{R}^{k+2}$  or  $D \subset \mathcal{L}^{k+2} \cup \mathcal{R}^{k+2} \cup D'$  which yields our claim. Replacing  $\Phi$  by  $D$  if necessary, we assume that both  $\mathcal{F} \not\subset \Phi$  and  $\mathcal{T} \not\subset \Phi$ .

Proposition 5.2 implies that for some  $l$  the intersections  $\Phi_n \cap \mathcal{F}_n$  and  $\Phi_n \cap \mathcal{T}_n \subset \mathcal{L}_n^l \cup \mathcal{R}_n^l$  for all  $n$ . Put  $d = \max(l, k+2)$ . Then  $\Phi_n \subset \mathcal{L}_n^d \cup \mathcal{R}_n^d$  and we have  $\Phi \subset \mathcal{L}^d \cup \mathcal{R}^d$ .  $\square$

*Proof of Theorem 1.6.* The theorem follows immediately from Propositions 5.6 and 5.9, Corollaries 5.3, 5.4 and 5.7, and Lemma 5.8.  $\square$

Let  $\Phi$  be an inductive system with  $\delta(\Phi) < p^{j+1}$  for some  $j \in \mathbb{Z}_{\geq 0}$ . Then each  $\phi \in \Phi_n$  can be uniquely represented in the form  $\otimes_{k=0}^j \phi_k^{[k]}$  with  $\phi_k \in \text{Irr}_p G_n$ . This notation is used in Proposition 5.10.

**Proposition 5.10** *Let  $\Phi$  be a BWM-system with  $\delta(\Phi) < p^{j+1}$ . Then there exists an integer  $N = N(\text{wdeg } \Phi, j)$  with the following properties: if  $d \geq N$ ,  $U_1, U_2 \subset \mathbb{N}_j$ ,  $U_1 \cap U_2 = \emptyset$ ,  $U_2 = \emptyset$  for  $p = 2$ ,  $\phi \in \Phi_n$ ,  $\phi_k \in \mathcal{F}_n$  for all  $k \in U_1$ ,  $\phi_k \in \mathcal{T}_n$  for all  $k \in U_2$ ,  $\phi_k \notin \mathcal{L}_n^d \cup \mathcal{R}_n^d$  for each  $k \in U_1 \cup U_2$ , and  $\phi \in \text{Irr}_n \psi$  with  $\psi \in \Phi_q$ ,  $q > n$ , then  $\psi_k \in \mathcal{F}_q$  for  $k \in U_1$ ,  $\psi_k \in \mathcal{T}_q$  for  $k \in U_2$ , and  $\psi_k \notin \mathcal{L}_q^d \cup \mathcal{R}_q^d$  for  $k \in U_1 \cup U_2$ .*

*Proof.* Let  $\text{wdeg } \Phi = c$ . Proposition 4.11 yields that for all  $a, b \in \mathbb{N}$  there exists  $t = t(a, b)$  such that the following holds: if  $n > t$ ,  $M = \otimes_{k=0}^s M_k^{[k]}$  with  $M_k \in \text{Irr}_p G_n$ , all  $M_k \in \mathcal{L}_n^a$  or all  $M_k \in \mathcal{R}_n^a$ ,  $\delta(M) \geq p^{s+1}$ ,  $F \in \mathcal{F}_n$  or  $\mathcal{T}_n$ , and  $F \notin \mathcal{L}_n^t$  or  $\mathcal{R}_n^t$ , respectively, then  $\text{wdeg}(M \otimes (F^{[s+1]})) > b$ . One may assume that  $t(a, b) \geq a + 2b$ . Now put

$$t_1 = t(c+2, c) \text{ and } t_u = t(t_{u-1}, c) \text{ for } 1 < u \leq j. \quad (19)$$

Hence

$$t_j > \dots > t_1 > c+2. \quad (20)$$

Set  $g = c+2 + \sum_{k=1}^j p^k t_k$ . Put  $N = \max(g, (c+1)^2 p^2 + 1)$ .

Let  $n > N$ ,  $\phi \in \Phi_n$  and satisfy the assumptions of the proposition with this  $N$  and some  $d \geq N$ . Assume that  $\psi \in \Phi_q$  and  $\phi \in \text{Irr}_n \psi$ . Arguing as in the proof of Proposition 5.9, one can conclude that

$$\psi_k \in \mathcal{L}_q^{c+2} \cup \mathcal{R}_q^{c+2} \cup \mathcal{F} \cup \mathcal{T} \quad (21)$$

for all  $k$ .

We claim that  $\phi_k \in \text{Irr}_n \psi_k$  for  $k \in U_1 \cup U_2$ . For  $k > 0$  and  $l < k$  put  $\pi(l, k) = \otimes_{s=l}^{k-1} \psi_s^{[s]}$ ,  $\pi(k) = \pi(0, k)$ ,  $\rho(l, k) = \otimes_{s=l}^k \psi_s^{[s]}$ , and  $\rho(k) = \rho(0, k)$ . Now our aim is to show that  $\delta(\pi(k)) < p^k$  if  $k \in U_1 \cup U_2$ . Assume this is false. If there exists  $i < k$  with  $\delta(\pi(i)) < p^i$ , choose maximal such  $i$  and put  $l = i$ . Otherwise put  $l = 0$ . Then  $\delta(\pi(l, k)) \geq p^k$ . One easily observes that  $\delta(\psi_l) \geq p$  since otherwise  $\delta(\pi(l+1)) < p^{l+1}$ , which contradicts the choice of  $l$ . Hence  $\omega(\psi_l) \notin \Omega_p(q)$ . So  $\psi_l \in \mathcal{L}_q^{c+2} \cup \mathcal{R}_q^{c+2}$  by (21). Assume that  $\psi_l \in \mathcal{L}_q^{c+2}$ . Put  $f_u = t_{u-l}$  for  $l < u \leq k$ . We claim that  $\psi_u \in \mathcal{L}_q^{f_u}$  for such  $u$ . Using (21), we conclude that  $\psi_u \in \mathcal{L}_q^{c+2} \cup \mathcal{R}_q^{c+2}$  if  $\omega(\psi_u) \notin \Omega_p(q)$ .

Recall that  $t_{u-l} \geq t_1 > c+2$ . First let  $u = l+1$ . As  $\text{wdeg } \rho(l, u) = \text{wdeg}(\psi_l \otimes (\psi_u^{[1]})) \leq c$ , we get  $\psi_u \in \mathcal{L}_q^{t_1}$  if  $\psi_u \in \mathcal{F}_q \cup \mathcal{T}_q$ . Since  $n > N > t_1 \geq 3c+2$ , Proposition 4.11 yields that  $\psi_u \notin \mathcal{R}_q^{c+2}$  if  $\omega(\psi_u) \neq 0$ . Hence  $\psi_u \in \mathcal{L}_q^{t_1}$ . Now assume that  $u > l+1$  and apply induction on  $u$ . Suppose that  $\psi_s \in \mathcal{L}^{f_s}$  for  $l < s < u$ . Then  $\psi_s \in \mathcal{L}^{f_{u-1}}$  for these  $s$  as  $f_s < f_{u-1}$  if  $s < u-1$ . The choice of  $l$  shows that  $\delta(\pi(l, u)) \geq p^u$  since otherwise  $\delta(\pi(u)) < p^u$ , which yields a contradiction. Write  $\rho(l, u) = \rho'^{[l]}$  and observe that  $\text{wdeg } \rho' = \text{wdeg } \rho(l, u) \leq c$ . Applying Proposition 4.11 and arguing as above, we conclude that  $\psi_u \notin \mathcal{R}_q^{c+2}$  and  $\psi_u \in \mathcal{L}_q^{t_{u-l}}$  if  $\psi_u \in \mathcal{F}_q \cup \mathcal{T}_q$ . Here it is essential that  $n > t_{u-l} \geq t_{u-1-l} + 2c$ . Put  $g' = c+2 + \sum_{h=1}^{k-l} p^h t_h$ . Then for  $u = k$  one has  $\rho' \in \mathcal{L}_q^{g'}$ . Obviously,  $g' \leq g$  (the equality holds only for  $l = 0$ ).

and  $k = j$ ). If  $\psi_l \in \mathcal{R}_q^{c+2}$ , similar arguments yield that  $\rho(l, k) = \rho'^{[l]}$  with  $\rho' \in \mathcal{R}_q^{g'}$ . Now we conclude that  $\otimes_{s=0}^{l-1} \phi_s^{[s]} \in \text{Irr}_n \pi(l)$  if  $l > 0$  and in all cases there exists  $\mu \in \text{Irr}_n \rho'$  with  $\mu = (\otimes_{s=0}^{k-1} \phi_{s+l}^{[s]}) \otimes (\chi^{[k-l+1]})$ ,  $\chi \in \text{Irr } G_n$ . Since  $\mathcal{L}^{g'}$  and  $\mathcal{R}^{g'}$  are inductive systems, this forces  $\phi_k \in \mathcal{L}_n^{g'}$  or  $\mathcal{R}_n^{g'}$  and yields a contradiction as  $\mathcal{L}^{g'} \cup \mathcal{R}^{g'} \subset \mathcal{L}^M \cup \mathcal{R}^M$ . Hence  $\delta(\pi(k)) < p^k$  if  $k > 0$  and  $k \in U_1 \cup U_2$ .

Now one can conclude that for all  $k \in U_1 \cup U_2$  there exists  $\mu \in \text{Irr}_n \psi_k$  with  $\mu = \phi_k \otimes (\mu'^{[1]})$ , where  $\mu' \in \text{Irr } G_n$  (this always holds for  $k = 0$ ). Obviously,  $\mu = \phi_k$  if  $\omega(\psi_k) \in \Omega_p(G_n)$ . Assume this is not the case. Then  $\psi_k \in \mathcal{L}_q^{c+2} \cup \mathcal{R}_q^{c+2}$  as we have seen earlier. But in this case  $\rho_k \in \mathcal{L}_n^{c+2} \cup \mathcal{R}_n^{c+2} \subset \mathcal{L}_n^M \cup \mathcal{R}_n^M$  which yields a contradiction. Hence  $\psi_k \in \mathcal{F}^q \cup \mathcal{T}^q$  and  $\phi_k \in \text{Irr}_n \psi_k$ . Naturally,  $\psi_k \notin \mathcal{L}_q^d \cup \mathcal{R}_q^d$  as otherwise  $\phi_k \in \mathcal{L}_n^d \cup \mathcal{R}_n^d$ . Now Lemmas 2.7 and 2.8 imply that  $\psi_k \in \mathcal{F}^q$  if  $\phi_k \in \mathcal{F}^q$  and  $\psi_k \in \mathcal{T}^q$  if  $\phi_k \in \mathcal{T}^q$ . This completes the proof.  $\square$

*Proof of Theorem 1.7.* Recall that an inductive system  $\Phi = \otimes_{k=0}^j \text{Fr}^k(\Phi^k)$  is special if each  $\Phi^k = C_L(a_1, \dots, a_d), C_R(a_1, \dots, a_d), \mathcal{F}$  or  $\mathcal{T}$ . If  $\Phi$  is special, we can write  $\Phi = \otimes_{f=0}^l \Psi^f$ , where  $\Psi^f$  are determined as before the statement of this theorem in Introduction. Theorem 4.10, Proposition 4.11, and Proposition 2.15 imply that a special inductive system  $\Phi$  is a BWM-system if and only if  $\delta(\Psi^f) < p^{i_f+1}$  for all  $f < l$  ( $i_f$  are such as in (11)). Lemma 5.1 yields that each special inductive system  $\Phi$  has the form  $\Phi = \langle \phi_n \mid n \geq A \rangle$  where  $\phi_n \in \text{Irr } G_n$ ,  $A \in \mathbb{N}$ . Hence special systems are indecomposable. Now we will show that every indecomposable BWM-system is a special system with  $\delta(\Psi^f) < p^{i_f+1}$  for  $f < l$ .

Let  $\Phi$  be an indecomposable inductive system and  $\text{wdeg } \Phi = c$ . By Theorem 1.5,  $\Phi = \otimes_{k=0}^j \text{Fr}^k(\Phi^k)$ , where  $\Phi^k$  are  $p$ -restrictedly generated inductive systems. It follows from Lemma 2.16 that  $\text{wdeg } \Phi^k \leq c$  for  $0 \leq k \leq j$ . One easily concludes that  $\Phi^k$  are indecomposable. By Theorem 1.6, each  $\Phi^k = C_L(a_1, \dots, a_d), C_R(a_1, \dots, a_d), \mathcal{F}$ , or  $\mathcal{T}$ , i.e.  $\Phi$  is special. This completes the proof of the theorem for indecomposable systems.

Next, let  $\mathcal{B}$  be an arbitrary BWM-system. Fix minimal  $j$  with  $\delta(\mathcal{B}) < p^{j+1}$ . Then for all  $n$  and each  $\phi \in \mathcal{B}_n$  we have  $\phi = \otimes_{k=0}^j \phi_k^{[k]}$  with  $\phi_k \in \text{Irr}_p G_n$ . Until the end of this proof for a module  $\psi \in \mathcal{B}_n$  we denote by  $\psi_k$ ,  $0 \leq k \leq j$ , the modules in  $\text{Irr}_p G_n$  that occur in such decomposition. Set

$$\Delta_{n,k} = \{M \in \text{Irr}_p G_n \mid M = \phi_k \text{ for some } \phi \in \Phi_n\}, \quad 0 \leq k \leq j.$$

Assume that  $\text{wdeg } \mathcal{B} = c$ . By Lemma 2.16,  $\text{wdeg } M \leq c$  for all  $M \in \Delta_{n,k}$ . Arguing as in the proof of Proposition 5.9, one concludes that

$$\Delta_{n,k} \subset \mathcal{F}_n \cup \mathcal{T}_n \cup \mathcal{L}_n^{c+2} \cup \mathcal{R}_n^{c+2} \quad (22)$$

for  $n > (l+1)^2 p^2$  and  $0 \leq k \leq j$ . First assume that

$$\text{for all } d \text{ there exist } n \text{ and } k \text{ with } \Delta_{n,k} \cap (\mathcal{F}_n \cup \mathcal{T}_n) \not\subset \mathcal{L}_n^d \cup \mathcal{R}_n^d. \quad (23)$$

If  $p \neq 2$ , denote by  $\mathcal{C}$  the collection of pairs  $(V_1, V_2)$ ,  $V_i \subset \mathbb{N}_j$  with the following properties:

- (i)  $V_1 \cap V_2 = \emptyset$ ,  $V_1 \cup V_2 \neq \emptyset$ ;
- (ii) for each  $d$  there exist  $n$  and  $\phi \in \mathcal{B}_n$  such that  $\phi_k \in \mathcal{F}_n$  for  $k \in V_1$ ,  $\phi_k \in \mathcal{T}_n$  for  $k \in V_2$ , and  $\phi_k \notin \mathcal{L}_n^d \cup \mathcal{R}_n^d$  for  $k \in V_1 \cup V_2$ ;
- (iii) there is no pair  $(V'_1, V'_2)$  such that  $V'_1$  and  $V'_2$  satisfy (i) and (ii),  $V_1 \subset V'_1$ ,  $V_2 \subset V'_2$  and  $V'_1 \cup V'_2 \neq V_1 \cup V_2$ .

If  $(V_1, V_2) \in \mathcal{C}$  and  $V_1 \cup V_2 = \mathbb{N}_j$ , set  $\Psi(V_1, V_2) = (\otimes_{k \in V_1} \text{Fr}^k(\mathcal{F})) \otimes (\otimes_{k \in V_2} \text{Fr}^k(\mathcal{T}))$ . Assume that  $(V_1, V_2) \in \mathcal{C}$  and  $V_1 \cup V_2 \neq \mathbb{N}_j$ . Set  $V_0 = \mathbb{N}_j \setminus (V_1 \cup V_2)$ . Fix  $t \in V_0$ .

The construction of  $\mathcal{C}$  implies that there exist  $u = u(t)$  and  $v = v(t)$  with the following properties: if  $\phi \in \Phi_n$ ,  $\phi_k \in \mathcal{F}_n$  for  $k \in V_1$ ,  $\phi_k \in \mathcal{T}_n$  for  $k \in V_2$ ,  $\phi_k \notin \mathcal{L}_n^u \cup \mathcal{R}_n^u$  for  $k \in V_1 \cup V_2$ , and  $\phi_t \in \mathcal{F}_n \cup \mathcal{T}_n$ , then  $\phi_t \in \mathcal{L}_n^v \cup \mathcal{R}_n^v$  (otherwise  $(V_1, V_2)$  would not satisfy (i)-(iii)). These arguments and Formulas (22) and (23) yield that there exist  $m$  and  $d$  such that  $\phi_k \in \mathcal{L}_n^d \cup \mathcal{R}_n^d$  if  $\phi \in \Phi_n$ ,  $n > (c+1)^2 p^2$ ,  $k \in V$ ,  $\phi_a \in \mathcal{F}_n \setminus (\mathcal{L}_n^m \cup \mathcal{R}_n^m)$  for all  $a \in V_1$ , and  $\phi_b \in \mathcal{T}_n \setminus (\mathcal{L}_n^m \cup \mathcal{R}_n^m)$  for all  $b \in V_2$ . Denote by  $S = S(V_1, V_2)$  the set of all inductive systems  $\Pi = \otimes_{k \in V_0} \text{Fr}^k(\Pi^k)$  with the following properties:  $\Pi^k = C_L(a_{1k}, \dots, a_{dk})$  or  $C_R(a_{1k}, \dots, a_{dk})$ ,  $0 \leq a_{ik} < p$ ,  $\Pi^k \subset \mathcal{L}^d$  or  $\mathcal{R}^d$ , and for each  $m$  there exist  $n$  and  $\phi \in \Phi_n$  with  $\phi_k = \rho_{n,L}(a_{1k}, \dots, a_{dk})$  or  $\rho_{n,R}(a_{1k}, \dots, a_{dk})$  if  $k \in V$  and  $\Pi^k \subset \mathcal{L}^d$  or  $\mathcal{R}^d$ , respectively,  $\phi_k \notin \mathcal{L}_n^m \cup \mathcal{R}_n^m$  for  $k \in V_1 \cup V_2$ ,  $\phi_k \in \mathcal{F}_n$  for  $k \in V_1$ , and  $\phi_k \in \mathcal{T}_n$  for  $k \in V_2$ . Observe that  $S$  is nonempty and finite. For  $\Pi \in S$  set

$$\Psi(\Pi) = \Pi \otimes (\otimes_{k \in V_1} \text{Fr}(\mathcal{F})) \otimes (\otimes_{k \in V_2} \text{Fr}(\mathcal{T})).$$

Put  $\Psi(V_1, V_2) = \cup_{\Pi \in S} \Psi(\Pi)$  and  $\Psi = \cup_{(V_1, V_2) \in \mathcal{C}} \Psi(V_1, V_2)$ . Proposition 5.2 implies that  $\Psi(V_1, V_2) \subset \mathcal{B}$  if  $V_1 \cup V_2 = \mathbb{N}_j$  and  $\Psi(\Pi) \subset \mathcal{B}$  for all  $\Pi \in S(V_1, V_2)$  if  $V_1 \cup V_2 \neq \mathbb{N}_j$ . Hence  $\Psi \subset \mathcal{B}$ .

For  $p = 2$  let  $\mathcal{C}$  be the collection of all nonempty sets  $V$  such that for each  $d$  there exist  $n$  and  $\phi \in \mathcal{B}_n$  with  $\phi_k \in \mathcal{F}_n$  for  $k \in V$  and  $V$  is a maximal subset in  $\mathbb{N}_j$  with this property. If  $\mathcal{C}$  consists of the set  $\mathbb{N}_j$ , put  $\Psi = \otimes_{k=0}^j \text{Fr}^k(\mathcal{F})$ . Assume this is not the case. For each  $V \in \mathcal{C}$  construct the set  $S(V)$  and the system  $\Psi(V)$  in the same way as we have constructed the sets  $S(V_1, V_2)$  and the systems  $\Psi(V_1, V_2)$  for  $p \neq 2$ . Put  $\Psi = \cup_{V \in \mathcal{C}} \Psi(V)$ . Using Proposition 5.2 as before, one concludes that  $\Psi \subset \mathcal{B}$  for  $p = 2$  as well. It is clear that in all cases  $\Psi$  is a finite union of indecomposable BWM-systems. So we are done if  $\Psi = \mathcal{B}$ .

Assume that  $\Psi \neq \mathcal{B}$  and set  $\mathcal{B}^1 = D(\mathcal{B}, \Psi)$ . Obviously,  $\text{wdeg } \mathcal{B}^1 \leq c$ . Denote by  $\Delta_{n,k}^1$  the analogues of the sets  $\Delta_{n,k}$  for the system  $\mathcal{B}^1$ . It is clear that (22) holds for  $\Delta_{n,k}^1$ .

Assume that (23) holds for  $\Delta_{n,k}^1$ . Then one can define the collection  $\mathcal{C}^1$  for the system  $\mathcal{B}^1$  in the same way as we have defined  $\mathcal{C}$  for  $\mathcal{B}$ . Put  $q(\mathcal{C}) = \max\{|V_1 \cup V_2| \mid (V_1, V_2) \in \mathcal{C}\}$  for  $p > 2$ ,  $q(\mathcal{C}) = \max\{|V| \mid V \in \mathcal{C}\}$  for  $p = 2$ , and define  $q(\mathcal{C}^1)$  similarly. We claim that  $q(\mathcal{C}^1) < q(\mathcal{C})$ . Indeed, let  $p > 2$  and  $(U_1, U_2) \in \mathcal{C}^1$ . We will show that there exists a pair  $(V_1, V_2) \in \mathcal{C}$  with  $U_i \subset V_i$  and  $|V_1 \cup V_2| > |U_1 \cup U_2|$ . First we will prove that  $(U_1, U_2) \notin \mathcal{C}$ . Suppose that  $(U_1, U_2) \in \mathcal{C}$  for some pair  $(U_1, U_2) \in \mathcal{C}^1$ . Let  $U_1 \cup U_2 \neq \mathbb{N}_j$ . We have constructed a subsystem  $\Psi(U_1, U_2) \subset \Psi$  such that for some  $m = m(U_1, U_2)$  if  $\phi \in \Phi_n$ ,  $\phi^k \in \mathcal{F}_n$  for all  $k \in U_1$ ,  $\phi^k \in \mathcal{T}_n$  for all  $k \in U_2$ , and  $\phi^k \notin \mathcal{L}_n^m \cup \mathcal{R}_n^m$  for each  $k \in U_1 \cup U_2$ , then  $\phi \in \Psi(U_1, U_2)_n$ .

Let  $N = N(c, j)$  be such as in Proposition 5.10. Let  $d \geq N$  if  $U_1 \cup U_2 = \mathbb{N}_j$  and  $d \geq \max\{N, m(U_1, U_2)\}$  otherwise. Since  $(U_1, U_2) \in \mathcal{C}^1$ , some  $\mathcal{B}_n^1$  contains a representation  $\phi$  such that  $\phi^k \in \mathcal{F}_n$  for  $k \in U_1$ ,  $\phi^k \in \mathcal{T}_n$  for  $k \in U_2$ , and  $\phi^k \notin \mathcal{L}_n^d \cup \mathcal{R}_n^d$  for each  $k \in U_1 \cup U_2$ . The construction of  $\mathcal{B}^1$  implies that for some  $t > n$  there exists a representation  $\rho \in \mathcal{B}_t \setminus \Psi_t$  with  $\phi \in \text{Irr}_n \rho$ . By Proposition 5.10,  $\rho_k \in \mathcal{F}_t$  for  $k \in U_1$ ,  $\rho_k \in \mathcal{T}_t$  for  $k \in U_2$ , and  $\rho_k \notin \mathcal{L}_t^d \cup \mathcal{R}_t^d$  for  $k \in U_1 \cup U_2$ . This yields a contradiction. Indeed, if  $U_1 \cup U_2 \neq \mathbb{N}_j$ , all such representations  $\rho \in \Psi(U_1, U_2)$  by the arguments above. If  $U_1 \cup U_2 = \mathbb{N}_j$ , the construction of  $\Psi(U_1, U_2)$  implies that for  $\rho \notin \Psi(U_1, U_2)$  some  $\rho_k \notin \mathcal{F}_t$  with  $k \in U_1$  or some  $\rho_s \notin \mathcal{T}_t$  for  $s \in U_2$ . Observe that in all cases  $\Psi(U_1, U_2) \subset \Psi$ . Hence  $(U_1, U_2) \notin \mathcal{C}$ .

The construction of  $\mathcal{C}$  and  $\mathcal{C}^1$  implies that the pair  $(U_1, U_2)$  satisfies the assumptions (i) and (ii) that we used to define  $\mathcal{C}$ , but does not satisfy (iii). Hence there exists a pair

$(U'_1, U'_2)$  mentioned in (iii).

Take for  $(V_1, V_2)$  such pair with maximal  $|U'_1 \cup U'_2|$ . For  $p = 2$  similar arguments yield that each  $U \subset \mathcal{C}^1$  is the proper subset of some  $M \subset \mathcal{C}$ . Hence in all cases  $q(\mathcal{C}^1) < q(\mathcal{C})$ .

Now construct an inductive system  $\Psi^1 \subset \mathcal{B}^1$  in the same way as  $\Psi$  was constructed for  $\mathcal{B}$ . If  $\Psi^1 \neq \mathcal{B}^1$ , set  $\mathcal{B}^2 = D(\mathcal{B}^1, \Psi^1)$ . Continue the process until this is possible, constructing for a system  $\mathcal{B}^i$  the collection  $\mathcal{C}^i$  and the subsystem  $\Psi^i$  in the same way as  $\mathcal{C}^1$  and  $\Psi^1$  were constructed. By the arguments above, if  $\mathcal{C}^i$  is determined, then  $q(\mathcal{C}^i) < q(\mathcal{C}^{i-1}) < \dots < q(\mathcal{C})$ . Hence for some  $i \leq j$  either  $\Psi^i = \mathcal{B}^i$  or (23) does not hold for  $\mathcal{B}^{i+1}$ . In the first case  $\mathcal{B} = \Psi \cup (\cup_{1 \leq k \leq i} \Psi^k)$  and hence is a finite union of indecomposable BWM-systems. Now assume that (23) does not hold for  $\mathcal{B}$  or  $\mathcal{B}^{i+1}$ . Set  $\Sigma = \mathcal{B}$  or  $\mathcal{B}^{i+1}$ , respectively. Formula (22) yields that  $\Sigma \subset \mathcal{L}^d \cup \mathcal{R}^d$  for some  $d$ . Hence  $\Sigma$  is a finite union of indecomposable BWM-systems by Proposition 5.6 and Lemma 5.8. This completes the proof.  $\square$

*Proof of Theorem 1.8.* Suppose that  $G_n = B_n(K)$ ,  $C_n(K)$  or  $D_n(K)$  and  $p$  satisfies the assumptions of the theorem. Throughout this proof we assume that  $n > 3$ . Set  $\mu_n = L(\omega_n^n)$  for  $G_n = B_n(K)$  or  $D_n(K)$  and  $\mu_n = L(\frac{p-1}{2}\omega_n^n)$  for  $G_n = C_n(K)$ ,  $\tau_n = L(0) \in \text{Irr } G_n$ , and  $\lambda_n = L(\omega_1^n)$  for all three types.

Let  $\Phi$  be a BWM-system. Theorem 1.2 yields that for  $n$  large enough the weight  $\omega(\varphi) \in \Omega(G_n)$  for every  $\varphi \in \Phi_n$ . Hence this holds for all  $n$  by Proposition 2.15. Lemma 3.1 implies that there exists  $l \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$  and each  $\varphi \in \Phi_n$  the representation  $\varphi = \bigotimes_{k=0}^l \varphi_k^{[k]}$  with  $\varphi_k \in \text{Irr}_p G_n$ ,  $0 \leq k \leq l$ .

For a triple of subsets  $A, B, C \subset \mathbb{N}_l$  such that  $A \cup B \cup C = \mathbb{N}_l$  and  $A \cap B = A \cap C = B \cap C = \emptyset$  put  $\psi_n(A, B, C) = \bigotimes_{k=0}^l \varphi_k^{[k]}$  with  $\varphi_k = \tau_n$  for  $k \in A$ ,  $\varphi_k = \lambda_n$  for  $k \in B$ , and  $\varphi_k = \mu_n$  for  $k \in C$ . Using Lemmas 2.10, 2.11, and 2.12, one easily observes that  $\psi_n(A, B, C) \in \text{Irr}_n \psi_{n+1}(A, B, C)$ . Hence the inductive system  $\Psi(A, B, C) = \langle \psi_n(A, B, C) \mid n > 3 \rangle$  is well defined. The same lemmas yield that

$$\Psi(A, B, C) = \bigotimes_{k=0}^l \text{Fr}^k(\Psi^k), \quad \Psi^k \in \{\mathcal{O}, \mathcal{L}, \mathcal{S}\}, \quad 0 \leq k \leq l$$

and that each inductive system

$$\Theta = \bigotimes_{k=0}^j \text{Fr}^k(\Theta_k), \quad \Theta_k \in \{\mathcal{O}, \mathcal{L}, \mathcal{S}\}, \quad 0 \leq k \leq j$$

coincides with  $\Psi(X, Y, Z)$  for some subsets  $X, Y, Z \subset \mathbb{N}_j$  such that  $X \cup Y \cup Z = \mathbb{N}_j$  and  $X \cap Y = X \cap Z = Y \cap Z = \emptyset$ . Hence all these systems  $\Theta$  are indecomposable.

Now we claim that for every  $\varphi \in \Phi_n$  there exists a triple  $A, B, C$  satisfying the assumptions of the previous paragraph such that

$$\psi_{n+1}(A, B, C) \in \Phi_{n+1} \quad \text{and} \quad \varphi \in \text{Irr}_n(\psi_{n+1}(A, B, C)). \quad (24)$$

Indeed, since  $\Phi$  is an inductive system, the representation  $\varphi \in \text{Irr}_n \chi$  for some  $\chi \in \Phi_{n+2}$ . One has  $\chi = \bigotimes_{k=0}^l \chi_k^{[k]}$  with  $\chi_k \in \Omega_p(G_{n+2})$ ,  $0 \leq k \leq l$ . It follows from Lemmas 2.10, 2.11, and 2.12 that  $\chi_k \in \mathcal{S}_{n+2}$  if  $\varphi_k \in \mathcal{S}_n$ ,  $\chi_k \in \mathcal{L}_{n+2}$  if  $\varphi_k \in \mathcal{L}_n$ , and  $\chi_k = \lambda_{n+2}$  for  $\varphi_k = \lambda_n$ . Then another application of those lemmas permits us to find such triple  $A, B, C$  that  $\psi_{n+1}(A, B, C) \in \text{Irr}_{n+1} \chi$  and  $\varphi \in \text{Irr}_n(\psi_{n+1}(A, B, C))$ . Naturally,  $\psi_{n+1}(A, B, C) \in \Psi_{n+1}$  as  $\Phi$  is an inductive system.

Define by  $I$  the collection of all such triples  $(A, B, C)$  that  $\psi_n(A, B, C) \in \Phi_n$  for all  $n$ . Observe that  $I \neq \emptyset$ . Set  $\Sigma = \bigcup_{(A, B, C) \in I} \Psi(A, B, C)$ . We claim that  $\Phi = \Sigma$ . Indeed,  $\Sigma \subset \Phi$  since  $\Phi$  is an inductive system. As the set of all systems  $\Psi(A, B, C)$  is finite and  $\psi_n(A, B, C) \in \Phi_n$  whenever  $\psi_{n+1}(A, B, C) \in \Phi_{n+1}$ , one easily concludes that there exists  $n_0$  such that for  $n > n_0$  the triple  $(A, B, C) \in I$  if  $\psi_n(A, B, C) \in \Phi_n$ . Now (24) yields that  $\Phi_n \subset \Sigma_n$  for  $n \geq n_0$ . Since  $\Sigma$  is an inductive system, this forces that  $\Phi \subset \Sigma$  and completes the proof.  $\square$

## References

- [1] Baranov, A.A.; Osinovskaya, A.A.; Suprunenko, I.D. Modular representations of classical groups with small weight multiplicities. *Journal of Mathematical Sciences* **161** (2009), 163–175.
- [2] Baranov, A.A.; Suprunenko, I.D. Branching rules for modular fundamental representations of symplectic groups. *Bull. London Math. Soc.* **32** (2000), 409–420.
- [3] Baranov, A.A.; Suprunenko, I.D. Minimal inductive systems of modular representations for naturally embedded algebraic groups of type  $A$ . *Communications in Algebra* **29** (2001), 3117–3134.
- [4] Baranov, A.A.; Suprunenko, I.D. Modular branching rules for 2-column diagram representations of general linear groups. *Journal of Algebra and Its Applications* **4** (2005), 489–515.
- [5] Benkart, G.; Britten, D.; Lemire, F. Modules with bounded weight multiplicities for simple Lie algebras. *Math. Z.* **225** (1997), no. 2, 333–353.
- [6] Britten, D.; Lemire, F. A classification of simple Lie modules having a 1-dimensional weight space. *Trans. Amer. Math. Soc.* **299** (1987), no. 2, 683–697.
- [7] Bourbaki, N. *Groupes et algèbres de Lie, Chaps. VII–VIII*; Hermann: Paris, 1975.
- [8] Carter R.W.; Cline, E. The submodule structure of Weyl modules for groups of type  $A_1$ , in *Proceedings of the Conference on Finite Groups (ed. W.R. Scott and F. Gross)*; Park City: Utah, 1975; Academic Press: New York/London, 1976, 303–311.
- [9] Fernando, S. L. Lie algebra modules with finite-dimensional weight spaces. I. *Trans. Amer. Math. Soc.* **322** (1990), no. 2, 757–781.
- [10] Fulton, W.; Harris, J. *Representation Theory: a First Course*; Springer-Verlag: New York, 1991.
- [11] Donkin, S. On tilting modules for algebraic groups. *Math. Z.* **212** (1993), 39–60.
- [12] Green, J.A. *Polynomial representations of  $GL_n$ , Lecture Notes in Math.* **830**; Springer: Berlin, 1980.
- [13] James, G.D. On the minimal dimensions of irreducible representations of symmetric groups. *Math. Proc. Camb. Phil. Soc.* **94** (1983), 417–424.

- [14] Jantzen, J.C. Darstellungen halbeinfacher algebraischer Gruppen und zugeordnete kontravariante Formen, *Bonner math. Schr.* **67** (1973).
- [15] Jantzen, J.C. *Representations of Algebraic Groups*; Academic Press: Orlando, 1987.
- [16] Kleschev, A.S. Branching rules for symmetric groups and applications. In Algebraic groups and their representations. NATO ASI Series. C **517** (1998), 103–130.
- [17] Kleschev A. On decomposition numbers and branching coefficients for symmetric and special linear groups. *Proc. Lond. Math. Soc.* **75** (1997), 497–558.
- [18] Mathieu, O. Classification of irreducible weight modules. *Ann. Inst. Fourier (Grenoble)* **50** (2000), no. 2, 537–592.
- [19] Osinovskaya, A.A. Restrictions of irreducible representations of classical algebraic groups to root  $A_1$ -subgroups. *Communications in Algebra* **31** (2003), 2357–2379.
- [20] Osinovskaya, A.A.; Suprunenko, I.D. Representations of algebraic groups of type  $D_n$  in characteristic 2 with small weight multiplicities, *Journal of Mathematical Sciences* **161** (2009), N 4, 558–564.
- [21] Osinovskaya, A.A.; Suprunenko, I.D. Representations of algebraic groups of type  $C_n$  with small weight multiplicities, *Journal of Mathematical Sciences* **171** (2010), N 3, 386–399.
- [22] Seitz, G.M. The maximal subgroups of classical algebraic groups. *Memoirs of the AMS* **365** (1987), 1–286.
- [23] Smith, S. Irreducible modules and parabolic subgroups. *J. Algebra* **75**, (1982) 286–289.
- [24] Suprunenko, I.D. On Jordan blocks of elements of order  $p$  in irreducible representations of classical groups with  $p$ -large highest weights. *J. Algebra* **191** (1997), 589–627.
- [25] Steinberg, R. Representations of algebraic groups. *Nagoya Math. J.* **22** (1963), 33–56.
- [26] Zalesskii, A.E. Group rings of locally finite groups and representation theory. In *Proceedings of the International Conference on Algebra*, Novosibirsk, 1989; *Contemporary Math.* **131** part 1, (1992), 453–472.
- [27] Zalesskii, A.E. Group rings of simple locally finite groups. In *Finite and locally finite groups*. NATO ASI Series. C **471** (1995), 219–246.
- [28] Zalesskii, A.E.; Suprunenko, I.D. Representations of dimensions  $(p^n \pm 1)$  of a symplectic group of degree  $2n$  over a finite field (in Russian). *Vestsi AN BSSR, Ser. Fiz.-Mat. Navuk*, no. 6 (1987), 9–15.
- [29] Zalesskii, A.E.; Suprunenko, I.D. Truncated symmetric powers of the natural realizations of the groups  $SL_m(P)$  and  $Sp_m(P)$  and their restrictions to subgroups. *Siber. Math. J.* **31**, no. 4 (1990), 555–566.